ON THE WEIERSTRASS APPROXIMATION THEOREM

BY

S. PASZKOWSKI (WROCŁAW)

This paper*) deals with several problems concerning the Weierstrass theorem for approximation with nodes[1]).

1. We introduce several notions and symbols. A closed interval I on the straight line is given. C denotes the class of continuous real-valued functions defined on I with the usual metric,

$$\|\xi-\eta\| = \max_{t\in I}|\xi(t)-\eta(t)|,$$

for any $\xi,\eta\in C$. A sequence of functions $\mu_0,\mu_1,\ldots,\mu_n,\ldots\in C$ is called a *Markoff sequence* if for any n the function

$$\sum_{k=0}^{n} a_k\mu_k \tag{1}$$

with

$$|a_0|+|a_1|+\ldots+|a_n|>0 \tag{2}$$

has at most n different roots in the interval I (see [1], p. 94). Functions of the size (1) without the condition (2) are called *polynomials* of the degree n. The set of all polynomials of this degree is denoted by W_n. The distance of a function $\xi\in C$ from the class W_n is defined by

$$\varepsilon_n(\xi) = \inf_{\psi\in W_n}\|\xi-\psi\|.$$

In the interval I a system T is given, consisting of m different numbers $t_1,t_2,\ldots,t_m$, called *nodes*. $W_n(\xi;T)$ denotes a subclass of W_n, consisting of all polynomials ω satisfying the conditions $\omega(t_i)=\xi(t_i)$ $(i=1,2,\ldots,m)$. For $n\geqslant m$ this class is not empty (see [2], p. 120, proof of theorem 1) and we shall suppose in the sequel that this inequality holds. The distance of the function $\xi\in C$ from the class $W_n(\xi;T)$ is defined by

$$\varepsilon_n(\xi,T) = \inf_{\omega\in W_n(\xi;T)}\|\xi-\omega\|.$$

2. THEOREM 1. *There exists a number $s>1$, depending only on the sets I and T, such that for any function $\xi\in C$ the following inequality holds:* $\varepsilon_n(\xi;T)\leqslant s\varepsilon_n(\xi)$.

Proof. Let ψ_n be a polynomial of the class W_n, which is nearest to ξ, *i. e.* such that $\|\xi-\psi_n\|=\varepsilon_n(\xi)$[2]).

The polynomials $\varphi_1,\varphi_2,\ldots,\varphi_m\in W_{m-1}\subset W_n$ will be determined by the conditions

$$\varphi_i(t_j)=\delta_{ij}\quad (i,j=1,2,\ldots,m), \tag{3}$$

where $t_1,t_2,\ldots,t_m$ are nodes. It is known that such a polynomial exists (see [1], p. 85-86, proof of the lemma). Let

$$\varphi=\sum_{i=1}^{m}\big(\xi(t_i)-\psi_n(t_i)\big)\varphi_i. \tag{4}$$

By (3) it follows that $\varphi(t_j)=\xi(t_j)-\psi_n(t_j)$ $(j=1,2,\ldots,m)$ and $\varphi+\psi_n\in W_n(\xi;T)$. Then by the definition of the number $\varepsilon_n(\xi;T)$ we have

$$\varepsilon_n(\xi;T)\leqslant\|\xi-(\varphi+\psi_n)\|\leqslant\|\xi-\psi_n\|+\|\varphi\|=\varepsilon_n(\xi)+\Big\|\sum_{i=1}^{m}\big(\xi(t_i)-\psi_n(t_i)\big)\varphi_i\Big\|$$

$$\leqslant\varepsilon_n(\xi)+\max_i|\xi(t_i)-\psi_n(t_i)|\cdot\Big\|\sum_{i=1}^{m}|\varphi_i|\Big\|\leqslant\Big(1+\Big\|\sum_{i=1}^{m}|\varphi_i|\Big\|\Big)\varepsilon_n(\xi).$$

Thus putting

$$s=1+\Big\|\sum_{i=1}^{m}|\varphi_i|\Big\| \tag{5}$$

we obtain theorem 1.

3. By theorem 1 the following inequalities hold:

$$\varepsilon_n(\xi)\leqslant\varepsilon_n(\xi;T)\leqslant s\varepsilon_n(\xi).$$

It shows that the relations

$$\lim_{n\to\infty}\varepsilon_n(\xi)=0, \tag{6}$$

$$\lim_{n\to\infty}\varepsilon_n(\xi;T)=0 \tag{7}$$

are equivalent. In the special case of

$$\mu_n=t^n\quad (n=0,1,\ldots) \tag{8}$$

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[1]) On other theorems concerning such approximation with nodes see my paper [2].

[2]) It is known that such a polynomial exists; see [1], p. 79.

relation (6) is the Weierstrass theorem; relation (7) follows also from the results of Yamabe [3]. But theorem 1 is stronger than (7). It is interesting to find such estimations for $\varepsilon_n(\xi; T)$ as are known for $\varepsilon_n(\xi)$ (for example if ξ is an analytic function). Theorem 1 enables us to obtain all those estimations and then gives a great deal of information on approximation with nodes.

4. Moreover, theorem 1 leads to other facts. Let us consider a modified approximation with nodes: with the increasing degree n of the polynomial, m_n, *i. e.* the number of nodes in T_n, increases. We confine ourselves to the Markoff sequence (8) generating algebraic polynomials. We assume of course that $m_n \leqslant n$. Furthermore, we assume that the smallest distance d_n between the points of T_n $(n = 1, 2, \ldots)$ satisfies the inequality $d_n \geqslant d|I|/m_n$, where d is a positive constant. Having made these assumptions we can prove the following theorem:

THEOREM 2. *If the sequence $\{m_n\}$ tends to infinity in such a way that*

$$m_n = o\left(\frac{|\log \varepsilon_n(\xi)|}{\log|\log \varepsilon_n(\xi)|}\right)^{3)},$$

then

$$\lim \varepsilon_n(\xi; T_n) = 0.$$

Proof. We note that if $\varepsilon_n(\xi) = 0$ for some n, then also $\varepsilon_n(\xi; T_n) = 0$ and the theorem is true for any $m_n \leqslant n$. In the sequel we assume that $\varepsilon_n(\xi) > 0$ for $n = 1, 2, \ldots$

In the case (8) expression (4) takes the form of the Lagrange interpolation formula

$$\varphi^{(n)}(t) = \sum_{i=1}^{m_n}\Bigg(\prod_{\substack{j=1\\ j\neq i}}^{m_n} \frac{t - t_j^{(n)}}{t_i^{(n)} - t_j^{(n)}}\Bigg)\cdot\big(\xi(t_i^{(n)}) - \psi_n(t_i^{(n)})\big),$$

where $\{t_1^{(n)}, t_2^{(n)}, \ldots, t_{m_n}^{(n)}\} = T_n$. Then by (5)

$$s - 1 = \max_{t \in I} \sum_{i=1}^{m_n} \prod_{\substack{j=1\\ j\neq i}}^{m_n} \left|\frac{t - t_j^{(n)}}{t_i^{(n)} - t_j^{(n)}}\right| \leqslant m_n \frac{|I|^{m_n - 1}}{\left(\dfrac{d|I|}{m_n}\right)^{m_n - 1}} = m_n \left(\frac{m_n}{d}\right)^{m_n - 1}.$$

For sufficiently great m_n, $m_n(m_n/d)^{m_n - 1} > 1$. Hence

$$s \leqslant m_n\left(\frac{m_n}{d}\right)^{m_n - 1} + 1 < 2m_n\left(\frac{m_n}{d}\right)^{m_n - 1} = 2d\left(\frac{m_n}{d}\right)^{m_n}. \tag{9}$$

[3]) $m_n = o(k_n)$ denotes that $\lim_{n\to\infty} (m_n/k_n) = 0$.

By hypothesis there exists a sequence $\{b_n\}$ such that

$$m_n = b_n \frac{|\log \varepsilon_n(\xi)|}{\log|\log \varepsilon_n(\xi)|} \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

By theorem 1 and (9) (for simplification we write ε_n for $\varepsilon_n(\xi)$)

$$\frac{\varepsilon_n(\xi; T_n)}{2d} < \left(b_n \frac{|\log \varepsilon_n|}{d \log|\log \varepsilon_n|}\right)^{b_n \frac{|\log \varepsilon_n|}{\log|\log \varepsilon_n|}} \varepsilon_n$$

and

$$\log \frac{\varepsilon_n(\xi; T_n)}{2d} < b_n \frac{|\log \varepsilon_n|}{\log|\log \varepsilon_n|} \log\left(b_n \frac{|\log \varepsilon_n|}{d \log|\log \varepsilon_n|}\right) + \log \varepsilon_n.$$

Considering then $\lim_{n\to\infty} \log \varepsilon_n = -\infty$, to prove $\lim_{n\to\infty} \log \varepsilon_n(\xi; T_n) = -\infty$, which implies the theorem, it is enough to verify that

$$\frac{b_n \dfrac{|\log \varepsilon_n|}{\log|\log \varepsilon_n|} \log\left(b_n \dfrac{|\log \varepsilon_n|}{d \log|\log \varepsilon_n|}\right)}{|\log \varepsilon_n|} \xrightarrow[n\to\infty]{} 0. \tag{10}$$

The expression on the left is equal to

$$\frac{b_n}{\log|\log \varepsilon_n|}\big(\log b_n + \log|\log \varepsilon_n| - \log(d \log|\log \varepsilon_n|)\big)$$

$$= \frac{b_n \log b_n}{\log|\log \varepsilon_n|} + b_n - \frac{b_n \log d}{\log|\log \varepsilon_n|} - \frac{b_n \log\log|\log \varepsilon_n|}{\log|\log \varepsilon_n|}.$$

The second member of this expression tends to 0 by definition, the first and the third member tend to 0 because of the relations

$$\lim_{n\to\infty} b_n \log b_n = 0, \quad \lim_{n\to\infty} \log|\log \varepsilon_n| = +\infty.$$

In the fourth member, for sufficiently great n,

$$0 < \log\log|\log \varepsilon_n| < \log|\log \varepsilon_n|.$$

Consequently, it also tends to 0, and hence relation (10) as well as theorem 2 are proved.

5. It is probable that theorem 2 may be strengthened, because it uses only a very rough estimation of the constant s.

Theorem 1 leads to the following problems, which are open also in the algebraic case (8):

P 163. Does there exist

$$\lim_{n\to\infty} \frac{\varepsilon_n(\xi;T)}{\varepsilon_n(\xi)}\ ?$$

(We know only that $1 \leqslant \varepsilon_n(\xi;T)/\varepsilon_n(\xi) \leqslant s < +\infty$ for any $n \geqslant m$.)

P 164. Is

$$\lim_{n\to\infty} \frac{\varepsilon_n(\xi;T)}{\varepsilon_n(\xi)} = 1\ ?$$

If the answer to both problems is positive, then

$$\lim_{n\to\infty} \frac{\varepsilon_n(\xi;T)}{\varepsilon_n(\xi)} = 1,$$

which would very essentially strengthen theorem 1.

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MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

ON THE NUMBER OF AFFINICALLY DIFFERENT SETS

BY

S. PASZKOWSKI (WROCŁAW)

The problem solved in this paper*) concerns the constructive theory of functions. In the theory of uniform approximation the notion of the *polynomial of the best approximation* of a continuous function $x(t)$ in the interval $\langle a, b\rangle$ plays an important role. Such a polynomial of degree n is defined in [1] as a polynomial realising

$$\min_{w\in W_n} \max_{a\leqslant t\leqslant b} |x(t)-w(t)|,$$

where W_n is the class of all polynomials of a degree not greater than n. Utilizing some results of de la Vallée Poussin [2], Rémès has given in [3] a recurrently defined sequence of polynomials, convergent to the polynomial of the best approximation. De la Vallée Poussin has remarked that the polynomial $v(t)$ of degree n which gives the best approximation of a function $x(t)$ on the system of $n+2$ points $t_0, t_1, \ldots, t_{n+1} (t_0 < t_1 < \ldots < t_{n+1})$ satisfies the system of $n+2$ equations

$$v(t_i)+(-1)^i\varepsilon = x(t_i) \qquad (i = 0, 1, \ldots, n+1)$$

(ε is here the $(n+2)$-th unknown). Thus, if $v(t) = a_0+a_1t+\ldots+a_nt^n$, then

$$a_0+a_1t_0+\ldots+a_nt_0^n+\varepsilon = x(t_0),$$

$$a_0+a_1t_1+\ldots+a_nt_1^n-\varepsilon = x(t_1),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_0+a_1t_{n+1}+\ldots+a_nt_{n+1}^n+(-1)^{n+1}\varepsilon = x(t_{n+1}).$$

Thus any coefficient of the polynomial $v(t)$ is a linear combination of the value of the function $x(t)$ with the coefficients depending only on the points $t_0, t_1, \ldots, t_{n+1}$:

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