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C O M M U N I C A T I O N S

# ON STATIONARY SEQUENCES OF RANDOM VARIABLES AND THE DE FINETTI'S EQUIVALENCE

BY

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In this paper we consider some fundamental properties of stationary sequences of random variables and of sequences of random variables equivalent in the sense of de Finetti.

All definitions, theorems and proofs are given by means of an uniform method, namely in terms of measures in the product-spaces in which some transformations take place.

In general the theorems proved in this paper are known: Theorems 4 and 5 present a paraphrase of the theorem on the decomposition of a measure-preserving transformation into indecomposable components (cf. *e. g.* [6], p. 242-244) and Theorems 1 and 6 are a paraphrase of Khintchin-Dynkin's theorem on random variables equivalent in the sense of de Finetti (see [1])[1]. Nevertheless, the equivalence of two definitions of symmetric measures (Theorem 6, $(\gamma)$) and the explicit use of the field of invariant sets (in the same theorem) seem to be new.

**1.** Let $X$ denote the product of a denumerable sequence of real axes. We denote a point of $X$ by $x = (x_1, x_2, \ldots)$, where $x_j$ are real. Let $\mathcal{B}$ denote the field of Borel subsets of $X$. We assume that all considered sets and functions are $\mathcal{B}$-measurable.

We shall frequently consider the functions defined in $X$, depending on some coordinates only, *e. g.* functions of the form $f(x) = f(x_1, x_2, \ldots, x_k)$. In such cases we shall treat the right side of the formula as a function defined for $x \in X$.

We denote by $\mathfrak{M}$ the class of all probability-measures in $\mathcal{B}$.

It is known that the investigation of sequences of random variables is equivalent to that of measures in product-spaces. Namely, by introducing the random variable $\xi(x) = x_n$ and putting for a certain $\mu \in \mathfrak{M}$

$$F_k(t_1, t_2, \ldots, t_k) = \mu\{x: x_1 < t_1, \ldots, x_k < t_k\},$$

we obtain $k$-dimensional distribution functions.

[1]) A new proof of the Khintchin-Dynkin theorem, based upon the methods of functional analysis, is due to Hewitt and Savage [3].

It is well-known that every sequence of distribution functions satisfying some compatibility conditions may be obtained in this way ([4], p. 27, Hauptsatz, and [2], § 49, Theorem A).

**2.** Let us denote by $\varphi$ the shift-transformation:

$$\varphi(\boldsymbol{x}) = (x_2, x_3, \ldots) \quad \text{for} \quad \boldsymbol{x} = (x_1, x_2, \ldots).$$

Obviously $\varphi^{-1}E \in \mathfrak{B}$ for $E \in \mathfrak{B}$.

We distinguish different subclasses of $\mathfrak{M}$.

Definition 1. A measure $\mu \in \mathfrak{M}$ is *stationary*, in symbols $\mu \in \mathfrak{M}_{st}$, if

(1) $$\mu(E) = \mu(\varphi^{-1}E) \quad \text{for} \quad E \in \mathfrak{B}.$$

Consequently,

(2) $$\mu(E) = \mu(\varphi^{-n}E) \quad (n = 1, 2, \ldots)$$

and

(3) $$\int f(\boldsymbol{x})\,d\mu = \int f(\varphi^n \boldsymbol{x})\,d\mu,$$

for $\mu$-integrable functions (where $\int \ldots d\mu$ always denotes the integral with respect to $\mu$ extended over $\boldsymbol{X}$).

Stationary sequences of random variables correspond to stationary measures in $\boldsymbol{X}$.

Definition 2. A set $E \in \mathfrak{B}$ is *invariant*, in symbols $E \in \mathfrak{B}_{inv}$, if $E = \varphi^{-1}E$.

Definition 3. A measure $\mu \in \mathfrak{M}$ is *indecomposable*, in symbols $\mu \in \mathfrak{M}_{ind}$, if $\mu(E) = 0$ or 1 for all $E \in \mathfrak{B}_{inv}$.

Definition 4. A measure $\mu \in \mathfrak{M}$ is *symmetrical*, in symbols $\mu \in \mathfrak{M}_{sym}$, if, for every sequence $n_1 < n_2 < \ldots$ of positive integers and every $\mu$-integrable function $f$, we have

(4) $$\int f(x_1, x_2, \ldots)\,d\mu = f(x_{n_1}, x_{n_2}, \ldots)\,d\mu.$$

It is possible to define the class $\mathfrak{M}_{sym}$ in another way. For each sequence $n_1 < n_2 < \ldots$ we consider the transformation

$$\psi(\boldsymbol{x}) = (x_{n_1}, x_{n_2}, \ldots)$$

and we require the invariance of the measure $\mu$:

(5) $$\mu(E) = \mu(\psi^{-1}E)$$

with respect to all $\psi$ of this form.

Putting $n_k = k+1$, we get $\mathfrak{M}_{sym} \subset \mathfrak{M}_{st}$.



We introduce temporarily one more definition:

Definition 4′. By replacing in definition 4 all increasing sequences of positive integers by all sequences of different positive integers, we obtain the notion of a *measure symmetrical in the strong sense.*

(It is easy to see, that in this definition we can consider only the permutations of the set of all positive integers.)

We shall prove that the two notions of symmetry are equivalent (Theorem 6 ($\gamma$)).

Sequences of random variables equivalent in the sense of de Finetti correspond to symmetrical measures.

**3.** Definition 5. $\mu$ is a *product measure*, in symbols: $\mu \in \mathfrak{M}_{pr}$, if it is stationary and

(6) $$\int f_1(x_1) f_2(x_2) \ldots f_k(x_k)\,d\mu = \int f_1(x_1)\,d\mu \int f_2(x_2)\,d\mu \ldots \int f_k(x_k)\,d\mu$$

for all $k = 1, 2, \ldots$ and all $\mu$-integrable $f_j$.

Obviously, it suffices to consider $f_j$ which are the characteristic functions of sets.

Condition (6) may also be replaced by the following:

(7) $$\int f_1(x_1, \ldots, x_k)\, g(x_{k+1}, \ldots, x_l)\,d\mu = \int f(x_1, \ldots, x_k)\,d\mu \int g(x_{k+1}, \ldots, x_l)\,d\mu$$

for all $\mu$-integrable $f$ and $g$.

The product measures satisfy not only (6) and (7), but also

(8) $$\int f(x_1)\,d\mu = \int f(x_n)\,d\mu \quad \text{for} \quad n = 1, 2, \ldots$$

A product measure is indecomposable (see [4], § 46, (3)). Product measures correspond to the sequences of independent and equidistributed random variables.

LEMMA 1. *Every stationary product measure is symmetrical in the strong sense.*

Proof. If $n_i \neq n_j$ (for $i \neq j$) and if the function $f$ is of the form

(9) $$f(\boldsymbol{x}) = g_1(x_1)\, g_2(x_2) \ldots g_k(x_k),$$

then, with respect to (6) and (8),

$$\int f(\boldsymbol{x})\,d\mu = \int g_1(x_{n_1})\,d\mu \ldots \int g_k(x_{n_k})\,d\mu = \int f(x_{n_1}, \ldots, x_{n_k})\,d\mu.$$

Thus, formula (4), which characterizes the symmetric measures, is proved for the functions $f$ of form (9). Since every $\mu$-integrable function may be approximated in the mean by the linear aggregates of the functions of that form, formula (4) is proved.

Lemma 1 permits the construction of a class of symmetrical measures.

LEMMA 2. *Let $\mu(E,t)$ be a function of two variables, $E\in\mathcal{B}$, and $t\in T$. Let $(T,\mathcal{T},\tau)$ be a probability-measure space. We suppose that*

*1° for any fixed $E\in\mathcal{B}$, the function $\mu(E,t)$ is $\mathcal{T}$-measurable,*

*2° $\tau$-a. e. (almost everywhere in $T$ with respect to $\tau$) the set function $\mu(E,t)$ is a probability measure symmetrical in the strong sense.*

*Then the set function*

$$\mu(E) \overset{\text{df}}{=} \int_T \mu(E,t)\,d\tau \quad \text{for} \quad E\in\mathcal{B} \tag{10}$$

*is symmetrical in the strong sense.*

Proof. If $f(\boldsymbol{x})$ is $\mu$-integrable, then, in view of the generalized Fubini theorem ([4], § 36, (3)), $f$ is integrable with respect to $\mu(\cdot,t)$ $\tau$-a. e. and

$$\int f(\boldsymbol{x})\,d\mu = \int_T d\tau \int_X f(\boldsymbol{x})\,d\mu(\cdot,t).$$

By 2° the measures $\mu(\cdot,t)$ are symmetrical in the strong sense $\tau$-a. e., whence

$$\int f(\boldsymbol{x})\,d\mu = \int_T d\tau \int_X f(x_{n_1},x_{n_2},\ldots)\,d\mu(\cdot,t) = \int f(x_{n_1},x_{n_2},\ldots)\,d\mu$$

for every sequence $n_1, n_2, \ldots$ of different positive integers.

Lemmas 1 and 2 imply

THEOREM 1. *If $\mu(E,t)$ satisfies condition 1° (see lemma 1) and the set function $\mu(\cdot,t)$ is a product measure $\tau$-a. e., then the measure $\mu$ defined by (10) is symmetrical in the strong sense.*

We shall prove (Theorem 6) that every symmetrical measure may be obtained in that way. Hence it follows that the converse of Theorem 1 is true and that symmetry and symmetry in the strong sense are equivalent.

**4.** THEOREM 2. *Every symmetrical and indecomposable measure is a product-measure: $\mathfrak{M}_{\text{sym}}\cdot\mathfrak{M}_{\text{ind}}\subset\mathfrak{M}_{\text{pr}}$.*

Proof. Let $f$ and $g$ be the characteristic functions of some sets of the form

$$f(\boldsymbol{x}) = f(x_1,x_2,\ldots,x_k), \qquad g(\boldsymbol{x}) = g(x_{k+1},x_{k+2},\ldots,x_l).$$

Since $\mu\in\mathfrak{M}_{\text{sym}}$, we have

$$\int f(\boldsymbol{x})g(\boldsymbol{x})\,d\mu = \int f(\boldsymbol{x})g(\varphi^n\boldsymbol{x})\,d\mu = \int f(\boldsymbol{x})\frac{1}{m+1}\sum_{n=0}^{m} g(\varphi^n\boldsymbol{x})\,d\mu \tag{11}$$

for $n, m = 0, 1, 2, \ldots$



Since $\mu\in\mathfrak{M}_{\text{st}}\cdot\mathfrak{M}_{\text{ind}}$, we obtain from the individual ergodic theorem (see *e. g.* [5])

$$\lim_{m\to\infty}\frac{1}{m+1}\sum_{n=0}^{m} g(\varphi^n\boldsymbol{x}) = \text{const} = \int g(\boldsymbol{x})\,d\mu \qquad \mu\text{-a. e.},$$

whence, in view of (11)

$$\int f(\boldsymbol{x})g(\boldsymbol{x})\,d\mu = \int f(\boldsymbol{x})\,d\mu \int g(\boldsymbol{x})\,d\mu.$$

This is formula (7), characterizing the product-measures among the stationary measures.

**5.** Let us introduce the conditional measure with respect to the field $\mathcal{B}_{\text{inv}}$ of all invariant sets.

For $\mu\in\mathfrak{M}$, for any fixed $E\in\mathcal{B}$, and for $Q$ running over $\mathcal{B}_{\text{inv}}$ the set function $\mu(EQ)$ is a (non-normed) measure, absolutely continuous with fespect to $\mu$ restricted to $\mathcal{B}_{\text{inv}}$. By the Radon-Nikodym theorem

$$\mu(EQ) = \int_Q \mu(E|\boldsymbol{x})\,d\mu, \tag{12}$$

where $\mu(E|\cdot)$ is $\mathcal{B}_{\text{inv}}$-measurable.

The function $\mu(E|\cdot)$ is uniquely determined modulo $\mu$. Since $\mathcal{B}$ is the field of all Borel sets, the function $\mu(E|\cdot)$ may be chosen so that ([2], § 48, (5))

(12′) $\mu(\cdot|\boldsymbol{x})$ is a probability measure $\mu$-a. e. in $\boldsymbol{X}$.

The following theorems concern the relations between the properties of $\mu$ and of $\mu(\cdot|\boldsymbol{x})$:

THEOREM 3. *If $\mu\in\mathfrak{M}_{\text{sym}}$, then $\mu(\cdot|\boldsymbol{x})\in\mathfrak{M}_{\text{sym}}$ $\mu$-a. e.*

THEOREM 4. *If $\mu\in\mathfrak{M}_{\text{st}}$, then $\mu(\cdot|\boldsymbol{x})\in\mathfrak{M}_{\text{st}}$ $\mu$-a. e.*

THEOREM 5. *If $\mu\in\mathfrak{M}_{\text{st}}$, then $\mu(\cdot|\boldsymbol{x})\in\mathfrak{M}_{\text{ind}}$ $\mu$-a. e.*

The proofs of these theorems will be based on the following lemmas.

LEMMA 3. *Let $\mathcal{B}_0$ be any finitely additive denumerable subfield of $\mathcal{B}$ with the following properties:*

(a) *The characteristic function of any set $E\in\mathcal{B}_0$ is of the form $\chi_E(\boldsymbol{x}) = \chi_E(x_1,\ldots,x_k)$ (where $k$ depends on $E$),*

(b) *$\mathcal{B}$ is the smallest $\sigma$-field containing $\mathcal{B}_0$.*

*Then, for any $\mu\in\mathfrak{M}$*

(i) *if $\mu(E) = \mu(\varphi^{-1}E)$ for all $E\in\mathcal{B}_0$, then $\mu\in\mathfrak{M}_{\text{st}}$;*

(ii) *if $\mu(E) = \mu(\psi^{-1}E)$ for all $E\in\mathcal{B}_0$ and for all $\psi$ (occuring in the definition of $\mathfrak{M}_{\text{sym}}$), then $\mu\in\mathfrak{M}_{\text{sym}}$.*

Proof is immediate.

Lemma 4. *There is a denumerable class* $\mathcal{C} \subset \mathcal{B}_{\rm inv}$ *such that, for any* $\mu \epsilon \mathfrak{M}_{\rm st}$, *we have*

(iii) *if* $\mu(E) = 0$ *or* 1 *for all* $E \epsilon \mathcal{C}$, *then* $\mu \epsilon \mathfrak{M}_{\rm ind}$.

Remark. Lemmas 3 and 4 state that it is possible to replace in the definition of $\mathfrak{M}_{\rm st}$, $\mathfrak{M}_{\rm sym}$ and $\mathfrak{M}_{\rm ind}$ the non-denumerable systems of equalities by some denumerable ones.

Proof of lemma 4. Class $\mathcal{C}$ will be defined as follows. Let $\chi_E(\boldsymbol{x})$ be a characteristic function of $E \epsilon \mathcal{B}_0$ (see lemma 3). It is easy to see that the sets

$$Q_{E,r} = \left\{\boldsymbol{x}: \overline{\lim_{m\to\infty}} \frac{1}{m+1}\sum_{n=0}^{m} \chi_E(\varphi^n \boldsymbol{x}) < r\right\} \qquad (r\text{— rational})$$

are invariant: $Q_{E,r} \epsilon \mathcal{B}_{\rm inv}$. We put $\mathcal{C} \overset{\rm df}{=} \{Q_{E,r}\}$.

We shall verify that $\mathcal{C}$ fulfills condition (iii). Let us suppose that $\mu \epsilon \mathfrak{M}_{\rm st}$ and $\mu(Q_{E,r}) = 0$ or 1. Consequently, for $E \epsilon \mathcal{B}_0$ we have

$$\overline{\lim_{m\to\infty}} \frac{1}{m+1}\sum_{n=0}^{m} \chi_E(\varphi^n \boldsymbol{x}) = \text{const} \qquad \mu\text{-a. e.} \tag{13}$$

On the other hand, by the individual ergodic theorem ($\mu \epsilon \mathfrak{M}_{\rm st}$) we have

$$\lim_{m\to\infty} \frac{1}{m+1}\sum_{n=0}^{m} \chi_E(\varphi^n \boldsymbol{x}) = g(\boldsymbol{x}) \qquad \mu\text{-a. e.}$$

and

$$\mu(E) = \int \chi_E(\boldsymbol{x})\,d\mu = \int g(\boldsymbol{x})\,d\mu.$$

Hence, formula (13) admits a sharper form:

$$\lim_{m\to\infty} \frac{1}{m+1}\sum_{n=0}^{m} \chi_E(\varphi^n \boldsymbol{x}) = \mu(E) \qquad \mu\text{-a. e.} \tag{14}$$

for all $E \subset \mathcal{B}_0$.

Let $F \epsilon \mathrm{B}_{\rm inv}$. We have to prove $\mu(F) = 0$, or 1. By the property (b) of $\mathcal{B}_0$, for each $\delta > 0$ there is a set $E \epsilon \mathcal{B}_0$ such that $\mu(E \dot{-} F) < \delta$.

Since $\chi_F(\boldsymbol{x}) = \chi_F(\varphi \boldsymbol{x})$, we have

$$\chi_F(\boldsymbol{x}) = \frac{1}{m+1}\sum_{n=0}^{m} \chi_F(\varphi^n \boldsymbol{x}),$$

and, consequently,

$$\int\left|\chi_F(\boldsymbol{x}) - \frac{1}{m+1}\sum_{n=0}^{m}\chi_E(\varphi^n \boldsymbol{x})\right|d\mu \leqslant \frac{1}{m+1}\sum_{n=0}^{m}\int|\chi_F(\varphi^n\boldsymbol{x}) - \chi_E(\varphi^n\boldsymbol{x})|\,d\mu < \delta.$$



Passing to the limit for $m \to \infty$, we obtain in view of (14)

$$\int|\chi_F(\boldsymbol{x}) - \mu(E)|\,d\mu < \delta.$$

It follows that $\chi_F(\boldsymbol{x}) = \text{const.}$ $\mu$-a. e. Hence $\mu(F) = 0$, or 1.

Proof of Theorem 3. Let us suppose that $\mu \epsilon \mathfrak{M}_{\rm sym}$. For fixed $E \epsilon \mathcal{B}_0$ and $Q \epsilon \mathcal{B}_{\rm inv}$ we have

$$\chi_E(\boldsymbol{x}) = \chi_E(x_1, \ldots, x_k), \qquad \chi_E(\psi \boldsymbol{x}) = \chi_E(x_{n_1}, \ldots, x_{n_k}),$$

and for all $l = 0, 1, 2, \ldots$

$$\chi_Q(\boldsymbol{x}) = \chi_Q(\varphi^l \boldsymbol{x}) = \chi_Q(x_{1+l}, x_{2+l}, \ldots).$$

Applying this formula twice, we obtain

$$\begin{aligned}\mu(EQ) &= \int \chi_E(x_1, \ldots, x_k)\chi_Q(x_{1+k}, x_{2+k}, \ldots)\,d\mu = \\ &= \int \chi_E(x_{n_1}, \ldots, x_{n_k})\chi_Q(x_{1+n_k}, x_{2+n_k}, \ldots)\,d\mu = \\ &= \int \chi_E(x_{n_1}, \ldots, x_{n_k})\chi_Q(x_1, x_2, \ldots)\,d\mu = \mu(\psi^{-1}E\cdot Q).\end{aligned}$$

Thus, in view of (12), we have

$$\int_Q \mu(E|\boldsymbol{x})\,d\mu = \int_Q \mu(\psi^{-1}E|\boldsymbol{x})\,d\mu.$$

The functions under the integral sign are $\mathcal{B}_{\rm inv}$-measurable, whence for each $E \epsilon \mathcal{B}_0$ and each $\psi$-

$$\mu(E|\boldsymbol{x}) = \mu(\psi^{-1}E|\boldsymbol{x}) \qquad \mu\text{-a. e.} \tag{15}$$

Since $\mathcal{B}_0$ is denumerable and satisfies (a), there is a common set $\boldsymbol{X}_0 \subset \boldsymbol{X}$ with $\mu(\boldsymbol{X}_0) = 0$ such that for $x \epsilon \boldsymbol{X} - \boldsymbol{X}_0$ equality (15) holds for all $E \epsilon \mathcal{B}_0$ and for all $\psi$. By Lemma 3 (ii) we obtain Theorem 3.

Proof of Theorem 4 is analogous: it suffices to replace $\psi$ by $\varphi$ in the preceding proof.

Proof of Theorem 5. Let $\mu \epsilon \mathfrak{M}_{\rm st}$ and $E \epsilon \mathcal{B}_{\rm inv}$. Hence

$$\mu(EQ) = \int_Q \chi_E(\boldsymbol{x})\,d\mu = \int_Q \mu(E|\boldsymbol{x})\,d\mu \qquad \text{for} \quad Q \epsilon \mathcal{B}_{\rm inv}.$$

The functions under the integral signs are equal $\mu$-a. e. since they are $\mathcal{B}_{\rm inv}$-measurable. Hence

$$\mu(E|\boldsymbol{x}) = 0 \text{ or } 1 \qquad \mu\text{-a. e.} \tag{16}$$

By Theorem 4, $\mu(\cdot|\boldsymbol{x}) \epsilon \mathfrak{M}_{\rm st}$ $\mu$-a. e. Consequently, there is a set $\boldsymbol{X}_0$ with $\mu(\boldsymbol{X}_0) = 0$ such that for $\boldsymbol{x} \epsilon \boldsymbol{X} - \boldsymbol{X}_0$ we have $\mu(\cdot|\boldsymbol{x}) \epsilon \mathfrak{M}_{\rm st}$ and at the same time formula (16) holds for all $E\epsilon\mathcal{C}$ (in view of the denumerability of $\mathcal{C}$). Lemma 4 gives $\mu(\cdot|\boldsymbol{x})\epsilon\mathfrak{M}_{\rm ind}$ for $\boldsymbol{x}\epsilon\boldsymbol{X}-\boldsymbol{X}_0$.

**6.** Combining Theorems 1, 2, 3 and 5 obtain the main result of this paper:

THEOREM 6. *Let $\mu$ be a symmetrical measure ($\mu\epsilon\mathfrak{M}_{\rm sym}$) and let $\mu(E|\boldsymbol{x})$ denote the conditional measure with respect to the field $\mathcal{B}_{\rm inv}$. Then:*

(α) *$\mu(\cdot|\boldsymbol{x})$ is a product measure $\mu$-a. e.,*

(β) *$\mu(E)=\int\mu(E|\boldsymbol{x})d\mu$ for $E\epsilon\mathcal{B}$,*

(γ) *$\mu$ is also symmetrical in the strong sense.*

Remark. Comparing Theorems 6 and 1, we see that $T=\boldsymbol{X}$, $\mathcal{T}=\mathcal{B}_{\rm inv}$ and $\tau=\mu|\mathcal{B}_{\rm inv}$. Thus, the construction in Theorem 6 is of inner character.

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*ZUR THEORIE DER LOKAL-KOMPAKTEN ABELSCHEN GRUPPEN*

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**Inhalt**



Die Theorie der lokal-kompakten abelschen Gruppen ist dank den bekannten Resultaten von Pontrjagin und van Kampen zu einem in Grundzügen fertiggebauten System geworden. Beiträge anderer Autoren haben weitere einzelne Ergebnisse gebracht, so daß hier scheinbar kein Forschungsgebiet mehr vorliegt. Dennoch lehrt eine Zusammenstellung der in der Literatur zerstreuten Resultate, daß gewisse Verknüpfungen zwischen Sätzen und Begriffen nach allem Anschein außer acht gelassen wurden und daß diesbezüglich neue Probleme auftauchen. Der Zweck dieser Arbeit ist somit, bekannte Eigenschaften von lokal-kompakten abelschen Gruppen auf die gegenseitige Abhängigkeit zu untersuchen, daraus einige neue Schlüsse zu ziehen und mehrere offene Probleme zu stellen. Es wird nicht vermieden auch bekannte Sätze neu zu beweisen, wo die Darstellung dabei an Einheitlichkeit gewinnen kann.

*EINLEITUNG*

Hier werden wir die Grundbegriffe erörtern. Als *topologische* (insb. *lokal-kompakte*) Gruppe bezeichnen wir eine Gruppe, deren Elemente einen topologischen (insb. lokal-kompakten) Raum bilden, in welchem die Operation $ab^{-1}$ stetig ist. Unter dem *topologischen* Raum ist hier eine Gesamtheit gemeint, in der eine Klasse von Untermengen bestimmt ist, die *Umgebungen* heißen und folgenden Bedingungen unterworfen sind: 1° jedes Element ist in einer Umgebung enthalten, 2° wenn ein Element