From (I) and (V) we obtain

$$\text{(VI)}\qquad \begin{aligned} z\in X\cdot &\equiv\cdot \sum_r \{E(r)=1\cdot M(r,z)\},\\ z\in -X\cdot &=\cdot \sum_r \{E(r)=0\cdot M(r,z)\},\end{aligned}$$

where

$$M(r,z)\cdot\equiv\cdot \sum_u \prod_t T(k_0,P(z,r),u,t)\cdot \prod_{x<r} \sum_u \prod_t T(k_1,P(z,x),u,t).$$

From (VI) we conclude that X and $-X$ belong to the class $\Sigma\Pi$.

Conversely, if $X,-X\in\Sigma\Pi$, then there exist two sets Z_1 and Z_2 such that

$$\begin{aligned} z\in X\cdot &\equiv\cdot \sum_v \sim P(z,v)\in Z_1,\\ z\in -X\cdot &\equiv\cdot \sum_v \sim P(z,v)\in Z_2.\end{aligned}$$

If one of the sets Z_1,Z_2 is empty, then also one of the sets $X,-X$ is empty, and X and $-X$ are computable. Hence e_X is also computable. If both the sets Z_1 and Z_2 are not empty, then they are recursively enumerable ones. Hence there are two computable functions f and g such that

$$\text{(VII)}\qquad \begin{aligned} z\in X\cdot &\equiv\cdot \sum_v \sim \sum_y P(z,v)=f(y),\\ z\in -X\cdot &\equiv\cdot \sum_v \sim \sum_y P(z,v)=g(y).\end{aligned}$$

Let U be the set of the values of the functions $2f$ and $2g+1$:

$$\text{(VIII)}\qquad x\in U\cdot\equiv\cdot \sum_y \{x=2f(y)\cdot\vee\cdot x=2g(y)+1\}.$$

From (VII) and (VIII) we obtain

$$\text{(IX)}\qquad \begin{aligned} z\in X\cdot &\equiv\cdot \sum_v \sim (2P(z,v)\in U)\cdot\equiv\cdot \sum_v e_U(2P(z,v))=0,\\ z\in -X\cdot &\equiv\cdot \sum_v \sim (2P(z,v)+1\in U)\cdot\equiv\cdot \sum_v e_U(2P(z,v)+1)=0.\end{aligned}$$

Also

$$\text{(X)}\quad e_X(z)=1-e_U(2P(z,(\min v)[e_U(2P(z,v))=0\cdot\vee\cdot e_U(2P(z,v)+1)=0])).$$

The function e_U satisfies the following condition:

$$\prod_z \sum_v \{e_U(2P(z,v))=0\cdot\vee\cdot e_U(2P(z,v)+1=0\}.$$

This follows at once from (IX). Therefore the operation of minimum used in (X) is effective, and the function e_X is computable with respect to the function e_U [5]).

[5]) An extension of the Theorem 2 was proved by S. C. Kleene: *Introduction to Metamathematics*, Amsterdam 1952, p. 293, Theorem XI.

SOME REMARKS ON PARTIALLY RECURSIVE FUNCTIONS

BY

A. JANICZAK †

EDITED BY A. GRZEGORCZYK *

The class $\boldsymbol{PR}$ of partially recursive functions is a natural generalization of the class of computable functions. In this note I shall mention some properties of the class $\boldsymbol{PR}$. Let $D^*(f)$ be the set of arguments and $D(f)$ the set of values of a function f. We shall consider only such functions for which both the sets $D^*(f)$ and $D(f)$ are not empty and are contained in the set of positive integers.

1. DEFINITION [1]). *$f\in\boldsymbol{PR}$ if and only if there exist two computable functions g and h such that*

$$\text{(1)}\qquad \prod_z z\in D^*(f)\cdot\equiv\cdot \sum_x h(z,x)=0,$$

$$\text{(2)}\qquad \prod_z z\in D^*(f)\cdot\supset\cdot f(z)=g((\min x)[h(z,x)=0]),$$

where $(\min x)[h(z,x)=0]=$ *the smallest x such that* $h(z,x)=0$.

The definition of partially recursive functions of many arguments is similar.

2. The above definition is equivalent to the following two:

$$f\in\boldsymbol{PR}\cdot\equiv\cdot \sum_g \{g\in\boldsymbol{R}\cdot D(g)=D^*(f)\cdot fg\in\boldsymbol{R}\},$$

$$f\in\boldsymbol{PR}\cdot\equiv\cdot D^*(f) \textit{ is a recursively enumerable set and } \prod_g \{g\in\boldsymbol{R}\cdot D(g)\subset D^*(f)\cdot\supset\cdot fg\in\boldsymbol{R}\},$$

where $\boldsymbol{R}$ is the class of computable functions, and fg is the superposition of the functions f and g.

3. There exists a function $f\in\boldsymbol{PR}$ which assumes only two values 0 and 1, and which cannot be extended to any computable function. This means that

$$\prod_g \{g\in\cdot\boldsymbol{R}\supset\cdot f\neq g|D^*(f)\}.$$

* See the footnote * on page 33.

[1]) See S. C. Kleene, *Recursive predicates and quantifiers*, Transactions of the Am. Math. Soc. 53 (1943), p. 41-73. The first definition was proposed by Kleene in the paper *On notation for ordinal numbers*, Journal of Symb. Logic 3 (1938), p. 152.

4. There exists a function $f \epsilon \boldsymbol{PR}$ which cannot be majorized by any function $g \epsilon \boldsymbol{R}$. This means that

$$\prod_g \{g \epsilon \boldsymbol{R} \cdot \supset \cdot \sum_x [x \epsilon D^*(f) \cdot f(x) > g(x)]\}.$$

5. If the set $D^*(f)$ is the complement of a recursively enumerable set, and f can be extended to a function $g \epsilon \boldsymbol{PR}$, then f can be extended to a function $h \epsilon \boldsymbol{R}$.

From this it follows that if $D^*(f)$ is a computable set and $f \epsilon \boldsymbol{PR}$, then f can be extended to a computable function.

6. The class $\boldsymbol{PR}$ is closed under the operation of substitution, but is not closed under the operation of minimum. Namely there exists a function $f \epsilon \boldsymbol{PR}$ such that setting $f_n(x) = f(n, x)$ we find that the function

$$g(n) = \textit{the smallest } x \textit{ such that } x \epsilon D^*(f_n) \textit{ and } f(n, x) = 0$$

is not partially recursive.

7. If $g, f \epsilon \boldsymbol{PR}$, X is a recursively enumerable set $X \subset D^*(f)$, and $X \subset D(g)$, then both the sets $f(X)$ and $g^{-1}(X)$ (image and counter-image of the set X) are recursively enumerable. In particular the set $D(f)$ is recursively enumerable provided that $f \epsilon \boldsymbol{PR}$.

Using partially recursive functions we can define in a very simple way some recursively enumerable sets. *E. g.* if $F_n(x)$ is the universal function for the class of primitive recursive functions and we put

$$f(n) = F_n((\min x)[F_n(x) > n^2]),$$

then the set $D(f)$ is a *simple set* (*i. e.* a recursively enumerable set with infinite complement which intersects all infinite recursively enumerable sets[2])).

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[2]) The definition and the first example of a simple set was given by Post. The example of Post is more complicated and obtained in a different way. See E. L. Post, *Recursively enumerable sets of positive integers and their decision problems*, Bulletin of the Am. Math. Soc. 50 (1944), p. 310.

ÜBER EINE EIGENSCHAFT DER SINGULÄREN KARDINALZAHLEN

VON

G. FODOR UND I. KETSKEMÉTY (SZEGED)

Aus einem Jourdainschen Satz[1]) kann folgende Behauptung hergeleiteten werd:

Wenn $\aleph_\alpha$ eine singuläre Kardinalzahl und ω_β die kleinste mit ω_α konfinale Anfangszahl ist, dann ist

$$\aleph_\alpha^{\aleph_\beta} > \aleph_\alpha.$$

In diesem Artikel geben wir einen direkten, einfachen Beweis dieser Behauptung an.

Beweis. E sei eine Menge von der Mächtigkeit $\aleph_\alpha$. Betrachten wir die Menge H sämtlicher Teilmengen von E, die die Mächtigkeit $\aleph_\beta$ haben. Offenbar ist $\overline{\overline{H}} = \aleph_\alpha^{\aleph_\beta}$. Nehmen wir an, dass $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha$ ist. Sei

$$(1) \qquad H_1, H_2, \ldots, H_\omega, H_{\omega+1}, \ldots, H_\xi, \ldots \qquad (\xi < \omega_\alpha)$$

eine wohlgeordnete Folge vom Typus ω_α sämtlicher Elemente von H. Sei y_1 ein beliebiges Element von H_1. Wählen wir aus H_2 ein Element y_2, das nicht y_1 gleich ist. Hat H_2 kein solches Element, so sei y_2 ein beliebiges Element von H_2. Nehmen wir an, dass wir y_ξ schon für jedes $\xi < \gamma < \omega_\alpha$ definiert haben. Dann sei y_γ ein von jedem y_ξ $(\xi < \gamma)$ verschiedenes Element von H_γ; wenn H_γ kein solches Element hat, sei y_γ ein beliebiges Element von H_γ. Setzen wir dieses Verfahren für jedes $\gamma < \omega_\alpha$ fort. Offenbar die Folge y_ξ enthält dann $\aleph_\alpha$ verschiedene Elemente.

Wir definieren eine wohlgeordnete Folge vom Typus ω_α

$$(2) \qquad x_1, x_2, x_3, \ldots, x_\omega, x_{\omega+1}, \ldots, x_\eta, \ldots \qquad (\eta < \omega_\alpha)$$

der Elemente von E folgendermassen. Sei $x_1 = y_1$. Haben wir schon jedes x_η für $\eta < \delta$ definiert, so definieren wir x_δ als das erste von allen diesen x_η verschiedene Element der Folge y_ξ.

Bezeichnen wir mit N die Menge derjenigen Elemente H_ξ der Folge (1), zu denen Elemente von (2) mit beliebig grossen Indizes gehören. Offensichtlich hat N die Mächtigkeit $\aleph_\alpha$. Jedem Element H_ξ von N sei nun $y_\xi = x_\mu$ zugeordnet. Aus der Konstruktion von (2) erhellt, dass diese Zuordnung eine eineindeutige Abbildung zwischen N und einer Teilmenge T von (2) von der Mächtigkeit $\aleph_\alpha$ ist.

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[1]) Ph. E. Jourdain, Quarterly Journal of Mathematics 39 (1908), p. 375-384.