Each of the 10 parts into which the class of all topological spaces is divided corresponds to one of the conjunctions $(*_*)$. The falsity of these conjunctions does not follow from $(**)$ since all the inclusions implied by $(**)$ are allowed for in the diagram. The parts marked by numbers (1)-(9) correspond to the respective cases (1)-(9) considered by Sikorski, whereas the part (0), corresponding to the conjunction

$$(0) \qquad (\overline{M})'(I)(D)',$$

has been left out by him.

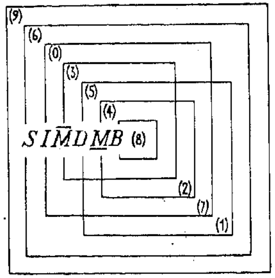

Fig. 1

To supplement the proof it is necessary to give an instance of a space possessing the property (0).

R. Sikorski has proved existence of a topological space $Y_1 \hat{+} Y_2$, possessing those and only those of the properties $(*)$ which are possessed by both spaces Y_1, Y_2.

One can easily verify, that if Y_1 fulfils conjunction (2) and Y_2 fulfils conjunction (3) (and the existence of such spaces is proved by Sikorski), $Y_1 \hat{+} Y_2$ possess the property (0). Thus Sikorski's proof is completed.

ON THE REDUCIBILITY OF DECISION PROBLEMS

BY

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EDITED BY A. GRZEGORCZYK*

The notion of general reducibility by means of machines, first mentioned briefly by Turing[1]), was defined precisely by Post[2]). Let "$X \to Y$" denote that the decision problem of the set X can be reduced to the decision problem of the set Y by means of Turing's machine.

In order to examine this relation we introduce the following definitions:

A function f is *computable with respect to a function* g if f belongs to the smallest class which:

1° contains the functions: g, $x+1$, $x+y$, $Ex = x - [\sqrt{x}]^2$;

2° is closed under the operations of substitution, of identification of variables, and of the effective minimum.

By the operation of *effective minimum* we mean the operation which leads from a function h to the function j defined as follows:

$$j(u) = (\min x)[h(u,x) = 0] = \text{the smallest } x \text{ such that } h(u,x) = 0$$

provided that the function h satisfies the condition

$$\prod_u \sum_x h(u,x) = 0$$

(if h does not satisfy this condition, then the operation $(\min x)$ is undefined).

Let e_X be the characteristic function of the set X. The function e_X assumes only the values $0,1$, and

$$\prod_z e_X(z) = 1 \cdot \equiv \cdot z \in X.$$

Using these notions we can prove the

THEOREM 1. *$X \to Y$ if and only if the function e_X is computable with respect to the function e_Y.*

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* This paper and the one immediately following are posthumous works of an author who died prematurely in Warsaw on July 5, 1951. The papers were prepared for publication from the notes left by the author.

[1]) See A. M. Turing, *Systems of logic based on ordinals*, Proc. of the London Math. Soc. 45 (1939), p. 173.

[2]) See E. L. Post, *Recursively enumerable sets of positive integers and their decision problems*, Bulletin of the Am. Math. Soc. 50 (1944), p. 311.

This theorem can be considered as a mathematical definition of general reducibility. It facilitates the proofs of many theorems concerning properties of the relation of general reducibility.

We shall consider the following example.

Let $\mathfrak{X}$ be the class of all sets X for which the decision problem is reducible to the decision problem of a recursively enumerable set:

$$X \epsilon \mathfrak{X} \cdot \equiv \cdot \sum_{Y} X \to Y \textit{ and } Y \textit{ is recursively enumerable.}$$

Let $\Sigma\Pi$ be the class of all sets X for which there exists a recursive relation R such that

$$x \epsilon X \cdot \equiv \cdot \sum_{y} \prod_{z} R(x,y,z).$$

Let $-X$ denote the complement of the set X.

THEOREM 2. *For every set X of positive integers, $X \epsilon \mathfrak{X}$ if and only if $X, -X \epsilon \Sigma\Pi$.*

Proof. We shall say that a function f is *recursive with respect to a function* g if the function f belongs to the smallest class which:

1° contains the function g, and $x+1$;

2° is closed under the operations of substitution, of identification of variables, and of recursion.

If the function e_X is computable with respect to the function e_Y, then the function e_X can be brought to the following *canonical form*:

$$\text{(I)} \qquad e_X(z) = E\big((\min x)[H_{k_0}(P(z,x)) = 0]\big);$$

here $H_k(u)$ is a universal function for the class of all functions which are recursive with respect to the function e_Y.

We assume that $H_0(u) = e_Y(u)$. $P(z,x)$ is a pairing function (*e. g.* $P(z,x) = 2^z(2x+1)-1$).

The function $H_k(u)$ can be defined by means of double recursion[3]). This recursive definition can be transformed in a familiar way into an arithmetical one by means of the double series of primes: $p(x,y) = p(P(x,y))$, where $p(x) =$ the x-th prime. The arithmetical definition of the relation $H_k(x) = z$ obtained in this manner has the form of an equivalence:

$$\text{(II)} \quad H_k(x) = z \cdot \equiv \cdot \sum_{v} \begin{cases} (1) \ \prod_{i \leqslant v} \exp(v, p(0,i)) > 0 \cdot \supset \cdot \exp(v, p(0,i)) = e_Y(i) + 1: \\ (2) \ \exp(v, p(k,x)) = z+1: \\ (3) \ R(v), \end{cases}$$

[3]) *E. g.* in a manner similar to that used by Robinson in his definition of a universal function for the class of primitive recursive functions. See R. M. Robinson, *Recursion and double recursion*, Bull. of the Am. Math. Soc. 54 (1948), p. 987-993.

where $\exp(v,x) = (\min t)\ [\sim(x^{t+1}|v)]$, and R is a recursive property which establishes certain relations between the (positive) exponents $\exp(v, p(n,i))$. Namely the relation R corresponds to the inductive step in the recursive definition of the function H. Condition (1) states that $H_0(x) = e_Y(x)$ and equation (2) that $H_k(x) = z$.

It follows from the conditions (1) and (3) that if $\exp(v, p(n,i)) > 0$, then $\exp(v, p(n,i)) = H_n(i) + 1$.

We also have the equivalence

$$\text{(III)} \qquad u = e_Y(i) + 1 \cdot \equiv : u = 2 \cdot \sum_{s} g(s) = i \cdot \vee \cdot u = 1 \cdot \prod_{t} g(t) \neq i,$$

where g is a computable function which enumerates the set Y.

From (II) and (III) it follows that the equivalence (II) can be written in the form

$$\text{(IV)} \qquad H_k(x) = z \cdot \equiv \cdot \sum_{v} \sum_{s} \prod_{t} S(v,s,t,k,x,z),$$

where S is a computable relation.

Indeed, substituting (III) for (I) we get an equivalence of the form

$$H_k(x) = z \cdot \equiv \cdot \sum_{v} \prod_{i \leqslant v} \sum_{s} \prod_{t} U(v,i,s,t,k,x,z)$$

with a recursive U. It is known[4]) that such a recursive V can be found that

$$\prod_{i \leqslant v} \sum_{s} \prod_{t} U(v,i,s,t,k,x,z) \equiv \sum_{s} \prod_{t} \prod_{i \leqslant v} V(v,i,s,t,k,x,z).$$

Taking

$$S(v,s,t,k,x,z) \equiv \prod_{i \leqslant v} V(v,i,s,t,k,x,z)$$

we obtain (IV) since the restricted quantifier $\prod_{i \leqslant v}$ does not lead beyond the class of computable relations.

From (IV), using the pairing functions, we obtain

$$\text{(V)} \qquad \begin{aligned} H_{k_0}(x) = 0 \cdot \equiv \cdot \sum_{u} \prod_{t} T(k_0, x, u, t), \\ H_{k_0}(x) \neq 0 \cdot \equiv \cdot H_{k_1}(x) = 0 \cdot \equiv \cdot \sum_{u} \prod_{t} T(k_1, x, u, t), \end{aligned}$$

where $H_{k_1}(x) = 1 - H_{k_0}(x)$ and T is a computable relation.

[4]) See A. Mostowski, *On a set of integers not definable by means of one-quantifier predicates*, Ann. Soc. Pol. Math. 21 (1948), theorem 3.3, p. 116.

From (I) and (V) we obtain

$$\text{(VI)}\qquad \begin{aligned} z \in X \cdot \equiv \cdot \sum_r \{E(r)=1 \cdot M(r,z)\}, \\ z \in -X \cdot = \cdot \sum_r \{E(r)=0 \cdot M(r,z)\}, \end{aligned}$$

where

$$M(r,z) \cdot \equiv \cdot \sum_u \prod_t T\big(k_0, P(z,r), u, t\big) \cdot \prod_{x<r} \sum_u \prod_t T\big(k_1, P(z,x), u, t\big).$$

From (VI) we conclude that X and $-X$ belong to the class $\Sigma\Pi$.

Conversely, if $X, -X \in \Sigma\Pi$, then there exist two sets Z_1 and Z_2 such that

$$z \in X \cdot \equiv \cdot \sum_v \sim P(z,v) \in Z_1,$$

$$z \in -X \cdot \equiv \cdot \sum_v \sim P(z,v) \in Z_2.$$

If one of the sets Z_1, Z_2 is empty, then also one of the sets $X, -X$ is empty, and X and $-X$ are computable. Hence e_X is also computable. If both the sets Z_1 and Z_2 are not empty, then they are recursively enumerable ones. Hence there are two computable functions f and g such that

$$\text{(VII)}\qquad \begin{aligned} z \in X \cdot \equiv \cdot \sum_v \sim \sum_y P(z,v) = f(y), \\ z \in -X \cdot \equiv \cdot \sum_v \sim \sum_v P(z,v) = g(y). \end{aligned}$$

Let U be the set of the values of the functions $2f$ and $2g+1$:

$$\text{(VIII)}\qquad x \in U \cdot \equiv \cdot \sum_y \{x = 2f(y) \cdot \vee \cdot x = 2g(y)+1\}.$$

From (VII) and (VIII) we obtain

$$\text{(IX)}\qquad \begin{aligned} z \in X \cdot \equiv \cdot \sum_v \sim \big(2P(z,v) \in U\big) \cdot \equiv \cdot \sum_v e_U\big(2P(z,v)\big) = 0, \\ z \in -X \cdot \equiv \cdot \sum_v \sim \big(2P(z,v)+1 \in U\big) \cdot \equiv \cdot \sum_v e_U\big(2P(z,v)+1\big) = 0. \end{aligned}$$

Also

$$\text{(X)}\qquad e_X(z) = 1 - e_U\big(2P(z, (\min v)[e_U(2P(z,v)) = 0 \cdot \vee \cdot e_U(2P(z,v)+1) = 0])\big).$$

The function e_U satisfies the following condition:

$$\prod_z \sum_v \{e_U(2P(z,v)) = 0 \cdot \vee \cdot e_U(2P(z,v)+1 = 0\}.$$

This follows at once from (IX). Therefore the operation of minimum used in (X) is effective, and the function e_X is computable with respect to the function e_U [5].

[5]) An extension of the Theorem 2 was proved by S. C. Kleene: *Introduction to Metamathematics*, Amsterdam 1952, p. 293, Theorem XI.

SOME REMARKS ON PARTIALLY RECURSIVE FUNCTIONS

BY

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The class $\boldsymbol{PR}$ of partially recursive functions is a natural generalization of the class of computable functions. In this note I shall mention some properties of the class $\boldsymbol{PR}$. Let $D^*(f)$ be the set of arguments and $D(f)$ the set of values of a function f. We shall consider only such functions for which both the sets $D^*(f)$ and $D(f)$ are not empty and are contained in the set of positive integers.

1. DEFINITION [1]). *$f \in \boldsymbol{PR}$ if and only if there exist two computable functions g and h such that*

$$(1)\qquad \prod_z z \in D^*(f) \cdot \equiv \cdot \sum_x h(z,x) = 0,$$

$$(2)\qquad \prod_z z \in D^*(f) \cdot \supset \cdot f(z) = g\big((\min x)[h(z,x) = 0]\big),$$

where $(\min x)[h(z,x)=0] \doteq$ *the smallest x such that* $h(z,x)=0$.

The definition of partially recursive functions of many arguments is similar.

2. The above definition is equivalent to the following two:

$$f \in \boldsymbol{PR} \cdot \equiv \cdot \sum_g \{g \in \boldsymbol{R} \cdot D(g) = D^*(f) \cdot fg \in \boldsymbol{R}\},$$

$$f \in \boldsymbol{PR} \cdot \equiv \cdot D^*(f) \textit{ is a recursively enumerable set}$$

$$\textit{and } \prod_g \{g \in \boldsymbol{R} \cdot D(g) \subset D^*(f) \cdot \supset \cdot fg \in \boldsymbol{R}\},$$

where $\boldsymbol{R}$ is the class of computable functions, and fg is the superposition of the functions f and g.

3. There exists a function $f \in \boldsymbol{PR}$ which assumes only two values 0 and 1, and which cannot be extended to any computable function. This means that

$$\prod_g \{g \in \cdot \boldsymbol{R} \supset \cdot f \neq g|D^*(f)\}.$$

* See the footnote * on page 33.

[1]) See S. C. Kleene, *Recursive predicates and quantifiers*, Transactions of the Am. Math. Soc. 53 (1943), p. 41-73. The first definition was proposed by Kleene in the paper *On notation for ordinal numbers*, Journal of Symb. Logic 3 (1938), p. 152.