Then putting $h_n=1-g_n$, we obtain (5). Consequently, denoting by $J$ the set of all sequences $\{j_n\}$ consisting of 0 and 1 and such that $j_n=h_n$ for sufficiently large $n$, we have

$$\mu\Big(\sum_{\{j_n\}\epsilon J} E_1^{j_1}E_2^{j_2}\ldots\Big)=\mu(\lim_{\overline{n}} E_n^{h_n})=1.$$

Now, let us denote by $J^*$ the set of all $\{j_n\}\epsilon J$ such that $\mu(E_1^{j_1}E_2^{j_2}\ldots)>0$. Since the set $J$ is denumerable, the preceding equality implies the following one:

$$\mu\Big(\sum_{\{j_n\}\epsilon J^*} E_1^{j_1}E_2^{j_2}\ldots\Big)=1.$$

It follows from the hypotheses, that every set belonging to $\boldsymbol{M}$ is the sum of a family of disjoint sets of the form $E_1^{j_1}E_2^{j_2}\ldots$, whence every set $E_1^{j_1}E_2^{j_2}\ldots$ with $\{j_n\}\epsilon J^*$ is an atom of $\mu$. Therefore, in view of the preceding formula, $\mu$ is purely atomic.

Combining (i) and (ii) we obtain

THEOREM IV. *There is a purely atomic measure $\mu$ and a sequence of stochastically independent sets $\{E_n\}$ with $\mu(E_n)=p_n$, if and only if*

(9) $$\sum_{n=1}^{\infty}(p_n)_*<\infty\ ^{9}).$$

The necessity of this condition follows directly from (i). To prove the sufficiency, it suffices to consider the Lebesgue measure in the unit interval, to apply 1(iv) and to define $\boldsymbol{M}$ as the $\sigma$-field spanned by $\{E_n\}$. It follows from (ii) that the Lebesgue measure, considered on $\boldsymbol{M}$ only, is purely atomic.

[9]) This construction of a sequence of sets independent with respect to a purely atomic measure was found also by S. Zubrzycki.

# ON THE AXIOMATIC TREATMENT OF PROBABILITY

BY

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The calculus of probability is a branch of mathematics whose foundations have so far not been fully investigated. There are of course, many such branches, but the calculus of probability is unique among them as regards the specific course of the development of its fundamental principles. This is bound with what prof. Steinhaus calls the "tavern" origin of the calculus of probability. A theory of gambling games at first, it gradually extended its range of applicability, becoming finally a mathematical theory of great practical and theoretical importance.

It was at a very early stage of the development of the calculus of probability that mathematicians felt the need of formulating its foundations more precisely. The first attempt in this direction was probably the definiton of "classical probability" given by Laplace. However it was the introduction of axiomatic methods, which made it possible to investigate the principles of probability along new lines.

The first axiomatic of probability was given by Bohlmann [2] about the year 1904. Since that time there have appeared (and still appear) numerous axiomatics, suggesting new methods of treatment or — more frequently — distorting treatments already known by means of the terminology which they adopt.

In principle it is the aim of every axiomatic of the calculus of probability to answer the following two questions:

1° What are events, *i. e.* what are those objects supposed to be probable?

2° What kind of function of events should probability be?

Rather a paradoxical point of view could be ventured, namely, that the answers to the above-mentioned questions should not be given by probabilists. The first should be answered by algebraists and the other by real function specialists.

And even if it were not true, experience shows that certain parts of algebra (lattice theory, and especially the theory of Boolean algebras) and certain parts of the theory of functions (measure theory) control the foundations of probability to such an extent that they almost absorb them. This is a useful process of complete mathematization of the calculus of probability.

The aim of this paper is to show, from the mathematical point of view, the present state of research regarding the axiomatic treatment of the calculus of probability.

It does not pretend to be complete. I omit all historical and philosophical problems connected with the subject, as well as many other problems based on strictly mathematical foundations of the calculus of probability and not directly connected with its axiomatic nor do I introduce any original suggestions; I simply try to compose various new results, particularly those obtained by Polish mathematicians.

I take into consideration the achievements directly connected with the axiomatic treatment of the calculus of probability and those which apparently are not connected with the axiomatic treatment but really influence it to a great extent.

**§ 1. What should an axiomatic of the calculus of probability be like?** From the unit circle (or from the interval [0,1]) I choose a point at random. What is the probability of this point belonging to a given set?

Such a problem leads at once to the consideration of Lebesgue's measure on the circumference of a circle (or in the interval [0,1]), and to the consideration of a field of measurable sets. Thus we have:

(a) a certain set $E$ (circumference of a circle, interval [0,1]),

(b) a certain field of sets $\boldsymbol{K}$ (measurable sets),

(c) a ceratin measure $\mu$ on $\boldsymbol{K}$ (Lebesgue's measure).

The occurence of the triplet $\langle E, \boldsymbol{K}, \mu\rangle$ is observed in nearly all problems of the calculus of probability and none of its elements is superfluous.

($a_0$) The set $E$, called the *set of elementary events*, is necessary to define the random variable as a real function on $E$ measurable with respect to the field $\boldsymbol{K}$.

($b_0$) The field $\boldsymbol{K}$, called the *field of events*, is the set of those objects which are supposed to be probable. In this field the set-theoretical operations correspond to the classical operations on events.

($c_0$) The measure $\mu$ on $\boldsymbol{K}$ is the probability attributed to the events of $\boldsymbol{K}$.

There are certainly many such triplets $\langle E, \boldsymbol{K}, \mu\rangle$ to which the probabilistic reasonings may refer. The aim of the axioms of probability is to select that class of them which is essential for probabilistic problems, *i. e.* triplets (we shall call them *spaces of probability*) with which the calculus of probability is and must be concerned if its applicability and results are not to be limited.



While selecting the spaces of probability from among the triplets $\langle E, \boldsymbol{K}, \mu\rangle$ we must proceed as follows:

(1.1) *Every such triplet that appears in the problems of the calculus of probability must be a space of probability (otherwise we shall limit the applicability of the calculus of probability).*

(1.2) *The fundamental notions of the calculus of probability should be definable for every space of probability (random variable, stochastical independence, mathematical expectation) and the fundamental theorems of the calculus of probability should be provable, for instance the laws of large numbers; otherwise we shall impoverish not only the theory but also its applicability.*

The above two conditions are of course not formulated exactly and may be interpreted in various ways. It seems that one condition which is a specification of the condition (1.1) might quite precisely be put as a condition for the spaces of probability. It runs as follows:

To the spaces of probability must belong the tripel consisting of the interval [0,1] (or the unit circle), of the field of measurable sets and of Lebesgue's measure on them.

Foundations of probability which exclude this case are only fragmentary and not interesting from the mathematical point of view.

**§ 2. Kolmogoroff's first interpretation — its merits and defects.** In the year 1933 A. N. Kolmogoroff published his work *Grundbegriffe der Wahrscheinlichkeitsrechnung* [10], in which he gave not only an axiomatic of the calculus of probability but also showed how it satisfies the postulates (1.1) and (1.2).

There is no need to emphasize here the decisive meaning of that work since in order to avoid redundancy we already adopted Kolmogoroff's standpoint in § 1.

However to all who know older textbooks and papers which deal whith the calculus of probability and know in what mistery different probabilistic notions were kept, it is clear that Kolmogoroff's work has indeed given mathematical foundations to this branch of knowledge. This has been achieved by an exact formulation of assumptions, a precise definition of notions and by establishment of close connection of the calculus of probability with other mathematical theories, namely the theory of measure and the theory of integral which were already fully developed in those days.

Kolmogoroff's axiomatic, according to the present terminology, demands that probability should be a normed measure (*i. e.* a non-negative and additive function) on a field $\boldsymbol{K}$ of subsets of $E$ satisfying the axiom of continuity:

$$\lim_{n\to\infty}\mu(X_n)=0 \quad \text{for} \quad X_1\supset X_2\supset \dots \text{ with } \prod_{n=1}^{\infty} X_n=0,$$

equivalent to the condition of denumerable additivity.

The axiom of continuity admits a unique extension of the measure $\mu$ to a denumerably additive measure on the smallest denumerably additive field of sets which contains the field $\boldsymbol{K}$.

Therefore we can always assume that $\boldsymbol{K}$ is already a denumerabl; additive field ($\sigma$-field) and that $\mu$ is a denumerably additive measure o1 $\boldsymbol{K}$ ($\sigma$-measure).

There is no need to discuss in detail how different probabilistic notions are defined on the ground of Kolmogoroff's axiomatic. It suffices to note that every real function $f$ on $E$ measurable with respect to the field $\boldsymbol{K}$ (hence such that $f^{-1}(A)\epsilon\boldsymbol{K}$ where $A$ is an arbitrary interval) is the random variable and the integral of $f$ on $E$ with respect to the measure $\mu$ is the expected value of the variable $f$. From the intuitive point of view the essence of Kolmogoroff's axiomatic is that only one kind of events are examined, namely those events which can be discribed as a random point which belong to a subset $X$ of $E$.

It appears that such an interpretation is always possible and often even necessary if we want to formulate certain problems (*e. g.* the existence of some stochastical processes).

On the other hand Kolmogoroff's axiomatic may be criticised. The author has done this himself [11].

The first objection concerns the representation of every event in the form $x\epsilon X$, which may be considered as an impoverishment of the formalism of the calculus of probability, or at least of its intuitive side and as deviation from its tradition. The second objection points out that his axiomatic does not admit the identification of almost identical events (*i. e.* such that their symmetrical difference is of measure 0), or which in fact is exactly the same, that, in most of the considered cases, it does not allow to introduce a strictly positive measure (*i. e.* a measure which is equal to 0 only on the empty set) These difficulties induced Kolmogoroff to suggest a somewhat different attitude towards the foundation of the calculus of probability; we shall discuss them in § 5.

**§ 3. Boolean algebras and fields of sets.** In general considerations upon the calculus of probability (*i. e.* those which do not refer to any special applications), with certain objects, which we call "events", we associate a certain number, which we call "probability".



It is essential that the nature of the events is, in this case, indifferent for a mathematician; it is necessary to furnish the set of events with the operations of addition, multiplication and complementation, governed by special laws.

From the formal point of view these laws are the same as those which govern the operations of addition, multiplication and complementation in the algebra of sets The above is not equivalent to regarding events as sets.

By a *Boolean algebra* we understand a class of objects furnished with operations governed by the same laws as the operations on sets.

The notion of Boolean algebra is of essential importance for the foundations of probab.lity. A set of events is a Boolean algebra. From the very definition of Boolean algebras in follows that the fields of sets are their particular case. M. H. Stone [22] has proved that also inversely: *every Boolean algebra is isomorphic with a field of sets.*

It is of great importance for the axiomatic treatment of the calculus of probability, as we shall see, that Stone's construction of a field of sets $\boldsymbol{K}$ isomorphic with a given Boolean algebra $\boldsymbol{B}$ has a clear probabilistic meaning and, moreover, it possesses certain properties which may be used in discussing the foundations of probability. This field is the field of both closed and open sets of some bicompact and totally disconnected topological space (the so-called *Boolean space*). The points of this space are the prime ideals of the algebra $\boldsymbol{B}$ (in this paper by *ideals* we understand the multiplicative ideals: $d$-ideals, and not the additive ideals: $s$-ideals).

Algebraically a *prime ideal* is defined as such a set of events $\boldsymbol{I}\subset\boldsymbol{B}$ that fulfils the following three conditions:

(3.1) $X\epsilon\boldsymbol{I}$, $Y\epsilon\boldsymbol{I}$ *implies* $X\cdot Y\epsilon\boldsymbol{I}$,

(3.2) $X\epsilon\boldsymbol{I}$, $Y\epsilon\boldsymbol{B}$ *implies* $X+Y\epsilon\boldsymbol{I}$,

(3.3) *from two complementary events $X$ and $X'$ one and only one belongs to* $\boldsymbol{I}$.

Suppose we carry out some trials on the occurence of a certain physical phenomenon; each trial gives a set of events which have occured in that case.

It is never one event, because from occurence of an event $X$ certainly follows the occurence of the event $X+Y$ ($Y$ is an arbitrary event) generally different from $X$. Therefore we see that this set fulfils the condition (3.2) of prime ideals. We can also verify that the remaining condi'ions are satisfied. We must only remember that the event $X+Y$ occurs if and only if at least one event $X, Y$ occurs; $X\cdot Y$ occurs if and only if both $X$ and $Y$

occur, and finally the event $X'$ occurs if and only if the event $X$ does not occur.

From the above it follows that the set of the events which occur in some trial is a prime ideal. This allows the prime ideals to be considered as elementary events.

The isomorphism, constructed by Stone, which maps the algebra of events $\boldsymbol{B}$ on a field of sets $\boldsymbol{K}$ situated in the space $S$ of all prime ideals has also a clear probabilistic meaning.

This mapping makes the event $X \in \boldsymbol{B}$ correspond to the set $\varphi(X)$, which consists of all those prime ideals $\boldsymbol{I}$ to which $X$ belongs. If we consider a prime ideal as a result of a trial, which is a substitute of an elementary event, then $\varphi(X)$ is the set of all those trials in which $X$ occurs. The mapping thus defined proves to be an isomorphism; the field $\boldsymbol{K}$ of all sets $\varphi(X)$ is isomorphic with $\boldsymbol{B}$, and if we accept the sets from $\boldsymbol{K}$ as neighbourhoods in $S$, then $S$ becomes a bicompact and totally disconnected topological space.

By *Boolean $\sigma$-algebra* we understand an algebra which, besides the operations discussed is furnished with the operation of denumerable addition and multiplication, *i. e.* the addition and multiplication is performable not only on two events but also on each denumerable sequence of events.

Similarly to the finite operations, also the infinite operations are governed by the same laws as the infinite operations on sets. Such a $\sigma$-algebra need not be isomorphic with a $\sigma$-field of sets.

However, Loomis and Sikorski [12,19] have shown that each Boolean $\sigma$-algebra is isomorphic with a *quotient $\sigma$-field, i. e.* a $\sigma$-field divided by a $\sigma$-ideal.

Here we shall shortly explain the operation of dividing an algebra by an ideal, in particular of a $\sigma$-algebra by a $\sigma$-ideal.

A subset $\boldsymbol{I}$ of a given algebra $\boldsymbol{B}$ is called an *ideal* if it satisfies the conditions (3.1) and (3.2) (if it also satisfies the condition (3.3) it is called a *prime ideal*). For instance the set of events of probability 1 is an ideal. In this interpretation the construction of quotient algebra can obtain a probabilistic intuition.

Let $\boldsymbol{I}$ be an ideal of the algebra of events $\boldsymbol{B}$; the elements of $\boldsymbol{I}$ will be called *certain events*. In the quotient algebra $\boldsymbol{B}/\boldsymbol{I}$ two events $X, Y \in \boldsymbol{B}$ whose simultaneous occurrence or non-occurrence is of certain 1 (*i. e.* the event $X \cdot Y$ belongs to $\boldsymbol{I}$) are treated as identical.

An ideal $\boldsymbol{I}$ is a *$\sigma$-ideal* (denumerably multiplicative ideal) if it satisfies additionally the following conditions:

(3.4) *If the events $X_1, X_2, X_3, \ldots$ belong to $\boldsymbol{I}$ then the product $\prod_{n=1}^{\infty} X_n$ also belongs to $\boldsymbol{I}$.*



If $\boldsymbol{I}$ is a $\sigma$-ideal of a $\sigma$-algebra $\boldsymbol{B}$, then the quotient algebra $\boldsymbol{B}/\boldsymbol{I}$ is also a $\sigma$-algebra.

**§ 4. Probability in Boolean algebras.** In order to avoid the difficulties discussed at the end of § 2 it will be convenient to omit the assumption that events supposed to be probable are sets and to assume only that they form a Boolean algebra. Such an attitude towards probability has been suggested by Glivenko [6] and Halmos [8]. Then probability is a normed measure on elements of a certain Boolean algebra, which may be arbitrary. In fact it is a kind of generalization of regarding events as sets; it is also a return to the classical traditions, according to which events need not be sets.

Let us note, however, that such an attitude deprives the fields of probability of one element, namely of the set $E$.

In a field of probability we shall now have only $\boldsymbol{B}$ and *i. e.* a Boolean algebra (analogous to a field of sets) and a measure, in place of the triplet $\langle E, \boldsymbol{K}, \mu \rangle$. This impoverishment causes difficulties in defining many probabilistic notions and in the first place in defining the random variable and its expected value.

Attempts have been made to eliminate this difficulty [1,3,18], but, in fact, they all reduce the notion of the random variable to the notion of a $\sigma$-homomorphism of a field of Borel sets of the real axis in a Boolean $\sigma$-algebra.

Such unification of these different attempts was first proposed by E. Marczewski and subsequently developed by Sikorski [20,21] and Götz [7].

Let $\mathfrak{B}$ be a field of Borel sets situated on the real axis. The mapping $h$ of $\mathfrak{B}$ into a Boolean algebra $\boldsymbol{B}$ is called a *homomorphism* if, for $A_1, A_2 \in \mathfrak{B}$

$$h(A_1 + A_2) = h(A_1) + h(A_2),$$
$$h(A_1 \cdot A_2) = h(A_1) \cdot h(A_2),$$
$$h(A_1') = h(A_1)'.$$

Moreover, if $\boldsymbol{B}$ is a $\sigma$-algebra and

$$h\Big(\sum_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} h(A_i)$$

for any $A_1, A_2, \ldots \in \mathfrak{B}$, then $h$ is called a *$\sigma$-homomorphism* (a denumerably additive homomorphism).

In order to show how the homomorphism of one field of Borel sets into a field of events may, for probabilistic purposes, replace the random variable, we shall consider a real function $f$ on the set $E$ measurable with respect to a certain $\sigma$-field $\boldsymbol{K}$ of the subset of $E$.

If $\boldsymbol{K}$ is regarded as a field of events, $f$ is regarded as the random variable (§ 1).

For an arbitrary Borel set $A$ on the real axis let us put

$$h(A)=\underset{x\in E}{E}[f(x)\in A]=f^{-1}(A).$$

Function $h$ is a $\sigma$-homomorphism of the field $\mathfrak{B}$ of Borel sets into $\boldsymbol{K}$. This homomorphism is strictly connected with the distribution function of $f$. If $A$ is the interval $[-\infty, a]$ and a $\sigma$-measure $\mu$ is given on $\boldsymbol{K}$, then putting $D_f(a)=\mu(h(A))$ we obtain the distribution function of $f$ with respect to the measure $\mu$.

In order to define Lebesgue's integral of the function $f$ with respect to the measure $\mu$, it is not necessary to know the function $f$ itself, it suffices to know its distribution function or the homomorphism $h$. This allows us to replace the notion of a measurable function by the notion of homomorphism in the foundations of probability constructed on Boolean algebras provided the algebra in question is a $\sigma$-algebra. In homomorphisms we obtain thus a substitute of the random variable. If we have a homomorphism $h$ of the field $\mathfrak{B}$ into a $\sigma$-algebra $\boldsymbol{B}$ furnished with a $\sigma$-measure $\mu$, then taking as example the definition of Lebesgue's integral, we can easily define the integral of the homomorphism $h$, which acts as the expected value and has all the properties generally demanded of the expected value.

**5. Making use of the connections of Boolean fields with fields of sets. Kolmogoroff's second interpretation.** The attitude described in the former paragraph is possible, but the same results may be obtained by making use of the connections between Boolean algebras and fields of sets.

We shall begin here with the description given by Kolmogoroff [11]. He remarked, above all, that it is easier to apply probability on Boolean algebras as it allows us to assume that probability is a strictly positive measure.

We obtain such a measure by identifying events with respect to the ideal of events of the probability 1.

He then remarked that in the case of Boolean algebras we need not assume the denumerable additivity of measure or the denumerable additivity of the algebra because for every Boolean algebra $\boldsymbol{B}_0$ with a strictly positive measure $\mu_0$ there exists a unique (with an exactitude to the isomorphism) $\sigma$-algebra $\boldsymbol{B}$ with a strictly positive $\sigma$-measure such that

*$\boldsymbol{B}$ is an extension of $\boldsymbol{B}_0$;*

*$\mu$ is an extension of $\mu_0$;*

*$\boldsymbol{B}$ itself is the least $\sigma$-subalgebra of $\boldsymbol{B}$ which contains $\boldsymbol{B}_0$.*



For a given algebra $\boldsymbol{B}_0$ with a strictly positive measure $\mu_0$ the algebra $\boldsymbol{B}$ and the measure $\mu$ are constructed as follows:

As can be seen from Stone's construction, the algebra $\boldsymbol{B}_0$ is isomorphic with a field of both closed and open sets $\boldsymbol{K}$ of a certain bicompact space $S$. By means of this isomorphism the measure $\mu_0$ may be transferred to $\boldsymbol{K}$. The measure $\mu_0$ in $\boldsymbol{K}$ satisfies the condition of continuity (see § 2), which follows easily from the bicompactness of the space $S$, and therefore it may be extended to a $\sigma$-measure on the least $\sigma$-field $\boldsymbol{K}_\beta$ which includes the field $\boldsymbol{K}$. The measure in $\boldsymbol{K}_\beta$ need not be strictly positive, whereas dividing $\boldsymbol{K}_\beta$ by the ideal of sets of the measure 1 we obtain a $\sigma$-algebra $\boldsymbol{B}$ and a $\sigma$-measure, thus satisfying the required conditions[1]). This not only allows us to omit the condition of denumerable additivity but also gives a convenient foundation for defining the random variables as functions on the space $S$, the elements of which, as we know, are the prime ideals in $\boldsymbol{B}_0$, and therefore treating them as elements is intuitively justified.

**§ 6. Detailed examination of homomorphisms.** The detailed examination of the connection between homomorphisms and point functions has been undertaken by Sikorski. The results he obtained seem to indicate that though probability may easily be examined on Boolean algebras, such treatment gives no generalizations. Any probabilistic examinations in algebras may be reduced to examinations in fields of sets.

Let $\boldsymbol{B}$ be a certain Boolean $\sigma$-algebra and $\mu$ a certain $\sigma$-measure in $\boldsymbol{B}$.

As we know, the algebra $\boldsymbol{B}$ is isomorphic with a certain field $\boldsymbol{K}$ of subsets of a fixed set $E$, divided by a $\sigma$-ideal $I$. Therefore $\boldsymbol{B}$ is isomorphic with $\boldsymbol{K}/\boldsymbol{I}$. The elements of $\boldsymbol{K}/\boldsymbol{I}$ we shall denote by $[X]$; it is the class of all those sets $Y\in\boldsymbol{K}$ which have been identified with $X\in\boldsymbol{K}$.

Let $f$ be a real function on $E$, measurable $\boldsymbol{K}$; putting for an arbitrary Borel set $A$

$$h(A)=\varphi([f^{-1}(A)]), \tag{6.1}$$

where $\varphi$ is an isomorphism mapping $\boldsymbol{K}/\boldsymbol{I}$ onto $B$, we obtain a homomorphism of the field of Borel sets $\mathfrak{B}$ in the algebra $\boldsymbol{B}$.

The homomorphism which satisfies the connection (6.1) for a certain function $f$ is called *induced by $f$*.

The consideration of induced homomorphisms can certainly be reduced to the consideration of inducing functions. If there existed homomorphisms not induced by functions, then considering them as if they

[1]) This construction differs from that of Kolmogoroff.

were random variables would effectively enrich the calculus of probability.

That however is not the case. Sikorski [20] has proved that every homomorphism of the field of Borel sets in a Boolean $\sigma$-algebra is induced by a certain function.

The expected value of homomorphisms can just as easily be reduced to the expected value (integral) of a point function. Namely, it should be noted that we can transfer the measure $\mu$ of $\boldsymbol{B}$ to the field $\boldsymbol{K}$ taking it through isomorphism to the algebra $\boldsymbol{K}/\boldsymbol{I}$, and then defining it on every set $X \epsilon \boldsymbol{K}$ as equal to $\mu([X])$. According to the measure thus obtained the sets belonging to $\boldsymbol{I}$ will be of the measure 1 (and if $\mu$ was a strictly positive measure in $\boldsymbol{B}$, then only these sets).

Now if $f_1$ and $f_2$ induce the same homomorphism $h$, then it is easy to see that

$$\underset{x \epsilon E}{E}[f_1(x)=f_2(x)]$$

belongs to $\boldsymbol{I}$, and therefore it is a set of the measure 1.

Thus it appears that functions which induce the same homomorphism are equivalent with respect to the measure in $\boldsymbol{K}$, and therefore particularly have the same integral with respect to this measure. Therefore, knowing the homomorphism $h$ we can define its expected value as an integral of an arbitrary function which induces this homomorphism.

Thus we shall obtain the same expected value of the homomorphism as that obtained through defining it directly, without the use of an inducing function (Sikorski [21]).

**§ 7. The algebra of sentences and their application in the axiomatic treatment of probability.** Tarski and Mostowski [23, 24, 17] have examined the connections between logic and Boolean algebras.

An algebra of sentences of a certain deductive theory forms a Boolean algebra if as the operations of addition, multiplication and complementation we take alternative, conjunction and negation, and furthermore if we identify two equivalent sentences. These connections allows us to perform the calculus of probability on sentences. which correpond to events.

These connections were of course known formerly, but not in a precise form. Keynes [9] was the first who purposely used sentences as elements with which probability is associated. It is also his merit that as early as 1921 he discovered and explained for probabilistic purposes, though not in a very exact way, a large part of the calculus of systems, which was afterwards developed by Tarski.



In Poland the calculus of probability on sentences was developed by Łukasiewicz [13] and Mazurkiewicz [15, 16].

Mazurkiewicz gave a most original axiomatic of the calculus of probability in which probability is a function not directly on sentences but on systems (*i. e.* on certain special sets of sentences). Systems, however, do not form a Boolean algebra but a more general algebraical structure — the so-called *Brouwerian algebra*. This is, I believe, the only attempt to develop the calculus of probability not in Boolean algebras. It was, however, given up by the author himself and not continued by others.

Practising the calculus of probability on sentences is a kind of specification of that method by which it is carried out on Boolean algebras. It consists in admitting only some special algebras. This has its good and bad points. From the intuitive point of view such procedure may be justified by the fact that the events considered in the calculus of probability are always described by means of certain sentences, the calculus in an algebra of events being the same as on the corresponding sentences in an algebra of sentences. Therefore one can identify events with sentences which describe them (or with the conjuction of those sentences).

In this way one obtains a convenient equivalent of events. The notion of ideal which, as we have seen, was of great importance as regards events, has also a very natural interpretation in the algebra of sentences. Ideals in the algebra of sentences are systems; prime ideals are complete systems [23].

The disadvantages of practising the calculus of probability on sentences, which discourage mathematicians from proceeding in this direction, are as follows:

1° Fields of sentences are always denumerable what implies that a Boolean algebra composed of sentences is at most denumerable. This renders it impossible to conduct the probabilistic investigation with regard to problems which require a non-denumerable field of events; in particular such a theory makes it impossible to reconstruct Lebesgue's measure on an interval (see § 1).

2° An algebra formed of sentences cannot be a $\sigma$-algebra, for it is easy to show that no denumerable Boolean algebra (*i. e.* of the power $\aleph_0$ exactly) is a $\sigma$-algebra.

3° The fact that events are sentences has never been properly utilized in the theory of probability built on sentences (with the exception of philosophical speculations of doubtful value).

The mathematical part of all the theories presented till now may be reconstructed without the assumption that elements supposed to be probable are sentences, if only we assume that they form a Boolean algebra.

Let us observe that when Kolmogoroff assumes that events are sets, he makes real use of it while defining the random variable. Similarly, in more specific questions of geometrical probability, if we assume that events are, for instance, sets on a plane, then we do so in order to make use of their geometrical properties. None of these assumptions would be appropriate if it were not used in an essential way.

The first two objections may be at least partially removed by applying to a Boolean algebra constructions given in the preceding paragraphs (which takes us beyond the algebra of sentences and, as far as I know, has not been worked out in detail by anybody); the last objection seems to be essential.

It is true that so far probabilistic logicians have not made sufficient use of the specific properties of their events. The latest work published by Carnap [5] may serve as example although it must be granted that in his previous work [4] he tried to use the special properties of sentences with regard to probabilistics.

Unfortunately the foundations of probability constructed by Carnap in [4] have so many drawbacks and are so fragmentary in their mathematical part (see for instance [14]) that they are of little value for mathematicians.

I suppose that it is not impossible to construct foundations of probability, based on sentences to which objection $3^0$ would not apply. One could even make use of certain ideas of Carnap concerning the invariations of probability with regard to permutations of signs ([4], p. 108). One could even expect certain connections with the theory of geometrical probability, in which appear invariations of probability with regard to the transformation group of a given geometry.

Even a superficial outline of such conceptions would lead far beyond the scope of this paper.

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