Theorem II enables us also to prove Littlewood's desideratum and to prove the following theorem: if $\varrho_0=\beta_0+i\gamma_0$ for $\beta_0\geqslant 1/2$ is a non-trivial zero of $\zeta(s)$, then for $T>c_{13}$ and $T>e^{c|\varrho_0|}$ we have

$$\max_{1\leqslant x\leqslant T}|\Delta(x)|>T^{\beta_0}e^{-c_{14}\frac{\log T\log\log\log T}{\log\log T}}.$$

Here c_{13} and c_{14} are numerical constants whose values can be given explicitly. To get finer results in this way we should need the one-sided refinement of our theorems mentioned in the first part.

The list of applications is still incomplete. But perhaps those discussed above already show that the way of interpretation of Dirichlet's and Kronecker's theorems which we have systematically followed, is a fruitful one. I hope I have also succeeded in showing that this theory is at the very beginning of its development and many more applications can be expected.

ON A THEOREM OF F. AND M. RIESZ

BY

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The theorem in question is the following ([3], or [4], p. 157-158): *Let*

$$f(re^{ix})=\sum_{n=0}^{\infty}a_n r^n e^{inx}$$

be an analytic function defined in the unit circle, and suppose that

$$\int_0^{2\pi}|f(re^{ix})|\,dx$$

is bounded for $r<1$. *Then there is a summable function* $f_0(e^{ix})$ *defined on the boundary of the circle and summable, such that*

$$\lim_{r\to 1}\int_0^{2\pi}|f_0(e^{ix})-f(re^{ix})|\,dx=0,$$

and the Fourier series of f_0 *is*

$$f_0(e^{ix})\sim\sum_{n=0}^{\infty}a_n e^{inx}.$$

The statement of the theorem and its original proof are function-theoretic. The purpose of this note is to give a new proof from a different point of view, which is closer to the spirit of some of the applications of the theorem[2]).

We shall have to consider bounded complex-valued measures defined on the field of Borel subsets of the interval $(0,2\pi)$. Associated with such a measure μ is a function of bounded variation on the interval defined by

$$\mu(x)=\mu(I_x),$$

where I_x is the set of y satisfying $0<y\leqslant x$. It will be clear from the context whether a symbol is being used to denote a measure or the corresponding function of bounded variation.

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[2]) The author expresses his indebtedness to Dr. S. Kakutani and Dr. J. Wermer for conversations about the subject of this note.

We are going to prove the Riesz theorem in this form ([2], p. 21):

Let μ be a bounded Borel measure on $(0, 2\pi)$ with Fourier-Stieltjes coefficients

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} d\mu(x) \qquad (n = 0, \pm 1, \ldots).$$

If $a_n = 0$ for all $n < 0$, then μ is absolutely continuous with respect to Lebesgue measure.

This assertion is equivalent to the Riesz theorem (see *e. g.* [4], p. 158 and p. 162).

LEMMA 1. *Under the hypothesis, μ vanishes on sets containing a single point.*

Indeed, decompose μ into a discrete part consisting of a sum of point masses, and a continuous part vanishing on sets of one point:

$$\mu = \mu_d + \mu_c.$$

For the corresponding Fourier-Stieltjes coefficients we have

$$a_n = a_n^d + a_n^c.$$

A well-known theorem of Norbert Wiener states that

$$\lim_{N\to\infty} \frac{1}{2N}\sum_{n=-N}^{N} |a_n^c|^2 = 0,$$

and so also

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=-N}^{0} |a_n^c|^2 = 0.$$

But $a_n^d + a_n^c = 0$ for $n < 0$, so that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=-N}^{0} |a_n^d|^2 = 0.$$

The coefficients of a discrete measure always form an almost-periodic sequence, so the last equation is impossible unless all the terms vanish. Hence μ is a continuous measure.

LEMMA 2. *Let*

$$\mu = \mu_s + \mu_a$$

be the Lebesgue decomposition of μ into a singular and an absolutely continuous part. Suppose φ is a continuous function of the form

$$\varphi(x) = \sum_{n=1}^{\infty} b_n e^{inx}, \qquad \sum |b_n| < \infty.$$

Then

$$\int \varphi(x)\, d\mu_s(x) = -\int \varphi(x)\, d\mu_a(x).$$

Since the series for φ converges absolutely, we can write

$$\int \varphi(x)\, d\mu(x) = \sum_{n=1}^{\infty} b_n \int e^{inx} d\mu(x) = \sum_{n=1}^{\infty} b_n a_{-n} = 0,$$

since a_n vanishes for negative indices. The fact that μ is orthogonal to φ is equivalent to the statement of the lemma.

Each of the measures μ_s and μ_a defines a linear functional on the space of all continuous functions on the boundary of the unit circle, and Lemma 2 shows that for a certain class of functions these functionals differ only in sign. We are to show that this impossible unless the singular measure vanishes.

LEMMA 3. *Denote by ν the total variation of μ_s:*

$$\nu(t) = \int_0^t |d\mu_s(x)|.$$

There is an analytic function $\varphi(z)$ in the unit circle having the properties

(a) $$0 < |\varphi(z)| < 1 \qquad (|z| < 1),$$

(b) $$\lim_{r\to 1} \varphi(re^{ix}) = 0 \quad a.\ e.\ (\nu).$$

Of course, φ can have radial limits equal to zero at most on a set of Lebesgue measure zero, but the exceptional set may carry all the mass of the singular measure ν, and this is the kind of φ we have to construct.

We form the non-negative harmonic function

$$u(r, x) = \frac{1}{2\pi}\int_0^{2\pi} P(r, x-y)\, d\nu(y),$$

where

$$P(r, x) = \frac{1-r^2}{1-2r\cos x + r^2}.$$

The classical theorem of Fatou states that $u(r, x)$ has radial boundary value $\nu'(x)$ at every point where this derivative exists. A modification of the proof of Fatou's theorem (as in [1], p. 148-151) shows that $u(r, x)$ tends to infinity at each point where $\nu'(x) = \infty$. We assert that $\nu(x)$ has infinite derivative almost everywhere (ν).

To see this, let $\gamma(t)$ be any function satisfying

$$\nu(\gamma(t)) = t \qquad (0 \leqslant t \leqslant \nu(2\pi)).$$

Thus γ is the inverse of ν at any point where that exists uniquely; at other points $\gamma(t)$ is to be chosen in the corresponding interval of constancy of ν. Then γ is a non-decreasing function, possibly having points of discontinuity. We use the same symbol to denote the associated measure on the interval $(0,\nu(2\pi))$. Evidently $\nu'(x)=\infty$ if $x=\gamma(t)$ and $\gamma'(t)=0$.

Now the variation of ν over an interval (α,β) is $\nu(\beta)-\nu(\alpha)$, or the Lebesgue measure of the image of (α,β) under the mapping ν. This property persists for arbitrary Borel sets. So to show that $\nu'(x)=\infty$ almost everywhere for the measure ν, it is enough to prove that $\gamma'(t)=0$ almost everywhere for Lebesgue measure.

Let F be the image under ν of the null-set on which the mass of ν is concentrated. Then F has full measure in $(0,\nu(2\pi))$; but evidently the variation of γ over F is zero. Thus the mass of γ is carried on the complement of F, which is a Lebesgue null-set, and the assertion is proved.

We have constructed a non-negative harmonic function $u(r,x)$ in the unit circle and proved that $u(r,x)$ has infinite radial boundary values except possibly in a null-set of ν. Let $v(r,x)$ be conjugate to $u(r,x)$, and form the analytic function

$$\varphi(re^{ix})=e^{-(u+iv)} \qquad (r<1).$$

Evidently φ satisfies the conditions of the lemma[3].

Define a function φ_r on the boundary of the circle by the formula

$$\varphi_r(e^{ix})=\varphi(re^{ix}) \qquad (r<1).$$

Then φ_r is continuous, has an absolutely convergent Fourier series, and its coefficients all vanish for negative indices. Indeed, if the power series expansion for φ is

$$\varphi(re^{ix})=\sum_{n=0}^{\infty}c_n r^n e^{inx} \qquad (c_0\neq 0),$$

then the same series is the Fourier series of φ_r. By Lemma 2,

$$\int e^{inx}\varphi_n(e^{ix})\,d\mu_s(x)=-\int e^{inx}\varphi_r(e^{ix})\,d\mu_a(x) \quad (r<1;n=1,2,\ldots).$$

As r increases to one, φ_r converges boundedly to zero almost everywhere for the measure μ_s. On the other hand, the Fatou theorem states that φ_r tends to a limit function almost everywhere for Lebesgue measure; call this limit $\varphi_0(e^{ix})$. Applying the Lebesgue convergence theorem to each integral, we obtain

$$\int e^{inx}\varphi_0(e^{ix})\,d\mu_a(x)=0 \qquad (n=1,2,\ldots).$$

[3]) The method used to construct φ is of course familiar.

Since φ never vanishes, $\varphi^{1/m}$ is analytic in the circle and has boundary values with the same properties as those of φ. Repeating the argument above, we have

$$\int e^{inx}\varphi_0^{1/m}(e^{ix})\,d\mu_a(x)=0 \qquad (m,n=1,2,\ldots),$$

where $\varphi^{1/m}$ is uniquely defined as

$$\varphi_0^{1/m}(e^{ix})=\lim_{r\to 1}e^{-(u+iv)/m}.$$

Evidently (recalling the construction of $u(r,x)$), we have for almost all x

$$\lim_{m\to\infty}\varphi_0^{1/m}(e^{ix})=1.$$

So letting m increase indefinitely and applying the Lebesgue convergence theorem once more, we obtain

$$\int e^{inx}\,d\mu_a(x)=0 \qquad (n=1,2,\ldots).$$

That is, the coefficients of μ_a vanish for negative indices, and consequently the same holds for μ_s.

LEMMA 4. *If the coefficients of a singular measure vanish for negative indices, then the measure is the null measure.*

This is of course a special case of the Riesz theorem. Making use again of the boundary properties of φ we can write

$$0=\lim_{r\to 1}\int\varphi_r(e^{ix})\,d\mu_s(x)=\lim_{r\to 1}\sum_{n=0}^{\infty}c_n r^n\int e^{inx}\,d\mu_s(x)=c_0a_0^s,$$

where a_n^s is a coefficient of μ_s. Since $c_0\neq 0$, we conclude that $a_0^s=0$. Continue by replacing $\varphi_r(e^{ix})$ by $e^{-ix}\varphi_r(e^{ix})$ in the same computations; we obtain $a_1^s=0$. Repeating the process, it appears that all the coefficients of μ_s vanish, and the lemma is proved.

We have shown that the singular part of $\mu=\mu_s+\mu_a$ vanishes and so μ is absolutely continuous, as was to be shown.

REFERENCES

[1] L. Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 2, Leipzig 1932.

[2] C.-G. Esseen, *Fourier analysis of distribution functions*, Acta Math. 77 (1945), p. 1-125.

[3] F. and M. Riesz, *Über die Randwerte einer analytischen Funktion*, Quatrième Congrès des math. scandinaves, 1916, p. 27-44.

[4] A. Zygmund, *Trigonometrical Series*, Warszawa-Lwów 1935.