ON THE CATEGORICITY IN POWER OF ELEMENTARY DEDUCTIVE SYSTEMS AND SOME RELATED PROBLEMS[1])

BY

J. ŁOŚ (TORUŃ)

A deductive system is *categorical* if it possesses only one model, in other words: if each two models of it are isomorphic. It is well known that no elementary system, which has an infinite model, is categorical[2]). Usually to prove this theorem, two models of different powers are constructed. It is evident that such two models may not be isomorphic. The problem arises, whether this theorem can be proved in a different way, *e. g.* by proving that for each elementary system which has an infinite model, there exist two non-isomorphic denumerable models.

The answer is: no. There are such elementary systems which have only one denumerable model. Such a system is called *categorical in the power* $\aleph_0$. In general we say that a system is *categorical in the power* $\mathfrak{m}$, if it possesses only one model of the power $\mathfrak{m}$.

In this paper I present some examples of systems which are categorical in different powers and some problems related to this notion, which appear to be interesting. I am not aware of any general theorem concerning this notion and I believe that those conjectured here are difficult to prove.

1. Notions. Let $r_1,\ldots,r_n$ be relation signs and $f_1,\ldots,f_k$ function signs and let ν_i and $\varkappa_i$ be numbers of arguments of r_i and f_i respectively. We call *elementary* a well-formed proposition in which appear only the signs $r_1,\ldots,r_n,f_1,\ldots,f_k$, individual variables: $x,y,z,\ldots$, logical sings: $\to$ (implication), $'$ (negation), $\vee$ (disjunction), $\wedge$ (conjunction), $\prod_x, \sum_x$ (quantifiers), $=$ (identity) and parenthesis. Therefore in an elementary proposition only variables of the lower type occur. A well-formed proposition in which variables of higher type occur (relation variables, function variables and so on) is called *non elementary*. By a *system* we understand here every consistent set of propositions such that every consequence of the propositions belonging to that set also belongs to that set. If $\mathfrak{X}$

[1]) Presented to the Polish Mathematical Society, Wrocław Section, on December 9, 1952.

[2]) See Th. Skolem, *Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen*, Fund. Math. 23 (1934), p. 150-161, „Bemerkungen der Redaktion", p. 161.

is an arbitrary consistent set of propositions, there exists a minimal system which contains $\mathfrak{X}$ and that system is said to be *generated* by $\mathfrak{X}$. If a system is generated by a set of elementary propositions, then it is called an *elementary system*.

A $(1+n+k)$-tupel $\mathfrak{M}=\langle A,R_1,\ldots,R_n,F_1,\ldots,F_k\rangle$, where A is a non-empty set, R_i a relation in A of ν_i arguments and F_i a function in A of $\varkappa_i$ arguments with values in A, is called a *model*. It is not necessary to determine the meaning of the notions of: validity in a model, isomorphism of models and submodels. These notions have a classical meaning.

By the *power of the model* $\mathfrak{M}$ is understood the power of A.

If $\mathfrak{X}$ is a set of well-formed propositions and each $\alpha\epsilon\mathfrak{X}$ is valid in $\mathfrak{M}$, then $\mathfrak{M}$ is called a *model of* $\mathfrak{X}$. If $\mathfrak{M}$ is a model of $\mathfrak{X}$ then $\mathfrak{M}$ is also a model of the whole system generated by $\mathfrak{X}$.

2. Examples. (2.1) If $n=1$, $\nu_1=2$, $k=0$ and $\mathfrak{X}_1$ consists of propositions

(i) $$[r_1(x,x)]',$$

(ii) $$[r_1(x,y)\wedge r_1(y,z)]\to r_1(x,z),$$

(iii) $$r_1(x,y)\vee r_1(y,x)\vee x=y,$$

then $\mathfrak{M}=\langle A,R_1\rangle$ is a model of $\mathfrak{X}_1$ if and only if R_1 orders A. In particular $\langle \boldsymbol{N},<\rangle,\langle \boldsymbol{R},<\rangle$, where $\boldsymbol{N}$ is the set of natural numbers and $\boldsymbol{R}$ the set of rational numbers, are models of $\mathfrak{X}_1$.

(2.2) If $n=0$, $k=1$, $\varkappa_1=2$ and $\mathfrak{X}_2$ consists of propositions

(iv) $$f_1(x,f_1(y,z))=f_1(f_1(x,y),z),$$

(v) $$\sum_x\prod_y f_1(x,y)=y,$$

(vi) $$\prod_x\prod_y\sum_z\sum_w[f_1(x,z)=y\wedge f_1(w,x)=y],$$

then $\mathfrak{M}=\langle A,F_1\rangle$ is a model of $\mathfrak{X}_2$ if and only if A is a group with multiplication (addition) F_1. In particular $\langle \boldsymbol{R},+\rangle$ is a model of $\mathfrak{X}_2$.

(2.3) Let $n=0,k=2,\varkappa_1=0,\varkappa_2=1$ (therefore f_1 is a constant individual sign). Let us denote by $\alpha(f_1,f_2)$, $\beta(f_2)$ and $\gamma(f_1,f_2)$ three propositions:

(vii) $$\prod_x (x=f_1)'\to\sum_y f_2(y)=x,$$

(viii) $$\prod_x\prod_y (f_2(x)=f_2(y)\to x=y),$$

(ix) $$\prod_\varrho \{[\varrho(f_1)\wedge\prod_x(\varrho(x)\to\varrho(f_2(x)))]\to\prod_y\varrho(y)\},$$

where ϱ is a variable relation (hence $\gamma(f_1,f_2)$ is not elementary), and by $\mathfrak{X}_3$ the set which consists of those three propositions. Evidently $\mathfrak{X}_3$ is the set of Peano's axioms of natural numbers, f_1 is the sign of 1 and f_2 the sign of the function $s^*=s+1$. It is well known that $\mathfrak{X}_3$ is categorical and therefore $\mathfrak{M}=\langle A,F_1,F_2\rangle$ is the model of $\mathfrak{X}_3$ if and only if $\mathfrak{M}$ is isomorphic with $\langle \boldsymbol{N},1,{}^*\rangle$.

(2.4) Now let the numbers n,k,ν_i and $\varkappa_i$ be arbitrary and $\pi_{\aleph_0}$ be the proposition

(x) $$\sum_z\sum_\varphi[\alpha(z,\varphi)\wedge\beta(\varphi)\wedge\gamma(z,\varphi)];$$

$\alpha(z,\varphi)$ arises from $\alpha(f_1,f_2)$ by replacing f_1 by z and f_2 by φ, where φ is a variable function. $\pi_{\aleph_0}$ is valid in $\mathfrak{M}=\langle A,R_1,\ldots,R_n,F_1,\ldots,F_k\rangle$ if and only if $\mathfrak{M}$ is of the power $\aleph_0$.

(2.5) Let $n=0, k=3, \varkappa_1=0, \varkappa_2=\varkappa_2=2$ and let $\mathfrak{X}_4$ be the set of all elementary propositions which are valid in the model $\langle \boldsymbol{N},1,+,\cdot\rangle$. Hence $\mathfrak{X}_4$ is the set of the so-called arithmetically true propositions. It is known that for each infinite cardinal number $\mathfrak{m}$, there exist two non-isomorphic models of $\mathfrak{X}_4$, both of power $\mathfrak{m}$[3]).

3. The notion of categoricity in power. A system is *categorical in the power* $\mathfrak{m}$, if it possesses a model of the power $\mathfrak{m}$, and if all its models of this power are isomorphic in pairs.

For some cardinal numbers $\mathfrak{m}$ this definition may be written in another form.

If for a cardinal number $\mathfrak{m}$, there exists a sentence $\pi_\mathfrak{m}$, which is valid in $\mathfrak{M}$ if and only if $\mathfrak{M}$ is of power $\mathfrak{m}$, then $\mathfrak{m}$ is called a *definable cardinal*. *E. g.* $\aleph_0$ is a definable cardinal (see example (2.4)). Each finite cardinal n is definable; π_n may be assumed to be elementary.

If $\mathfrak{m}$ is a definable cardinal, then the system generated by $\mathfrak{X}$ is categorical in the power $\mathfrak{m}$ if and only if the system generated by $\mathfrak{X}+(\pi_\mathfrak{m})$ is categorical.

Some necessary conditions for the categoricity in the power of elementary systems are obvious.

An elementary system X is *complete*, if for each elementary proposition α, either $\alpha\epsilon\mathfrak{X}$, or $\mathfrak{X}+(\alpha)$ is an inconsistent set. It is obvious that

(3.1) If the elementary system $\mathfrak{X}$ is categorical in the power $\mathfrak{m}\geqslant\aleph_0$ and if it has no finite models, then $\mathfrak{X}$ is complete.

(3.2) If the elementary system $\mathfrak{X}$ is categorical in the power $n<\aleph_0$ then the system generated by $\mathfrak{X}+(\pi_n)$ is complete.

From the well-known theorems on extension of models of elementary systems it follows that:

[3]) See Th. Skolem, l. c.

(3.3) If $\mathfrak{M}$ is a model of the power $\mathfrak{m}\geqslant\aleph_0$, of an elementary system which is categorical in the power $\mathfrak{m}$, then for each ordinal number ξ of power $\leqslant\mathfrak{m}$, there exists an increasing ξ-sequence of submodels of $\mathfrak{M}$ and such that each submodel of that sequence is isomorphic with $\mathfrak{M}$.

4. Examples of systems categorical in different power. Let $\mathfrak{X}_5$ consist of sentences (i), (ii), (iii) and the two sentences given below:

$$\prod_x\sum_y\sum_z(r_1(y,x)\wedge r(x,z)),$$

$$\prod_x\prod_y[r_1(x,y)\rightarrow\sum_z(r_1(x,z)\wedge r_1(z,y))].$$

The sentences $\mathfrak{X}_5$ are axioms of the dense order without the first and the least elements. Therefore from Cantor's characteristic of the order type η it follows that each denumerable model $\mathfrak{M}=\langle A,R_1\rangle$ of $\mathfrak{X}_5$ is isomorphic with $\langle \boldsymbol{R},<\rangle$, and thus the system generated by $\mathfrak{X}_5$ is categorical in the power $\aleph_0$.

Another example of a system which is categorical in the power $\aleph_0$ is the system of atomfree Boolean algebras. The axioms of this system are written in an elementary way, hence it is also elementary. It is known that every denumerable, atomfree Boolean algebra is isomorphic with the algebra of both closed and open sets of Cantor's discontinuum.

Evidently, both the above mentioned systems are categorical only in the power $\aleph_0$, but there exist systems which are categorical in every infinite power.

Let $\mathfrak{X}_6$ be the set of axioms of groups (propositions (iv), (v), (vi), ex. (2.2)) with an additional axiom which indicates that every element is of the range p, where p is a prime number. This axiom may be written, for instance for $p=3$,

$$\prod_x\prod_y\{y=f_1(f_1(x,x),x)\rightarrow f_1(y,y)=y\},$$

and for other prime numbers p as well.

The system generated by $\mathfrak{X}_6$ is elementary and categorical in every power in which it has a model. It is well known that a group every element of which is of the range p, is a linear space upon the field of integers mod p. As linear space that group has a basis, whose power depends only on the power of the group. But two linear spaces with basis of the same power are isomorphic.

And here is an example of an elementary system which is categorical in each power $\mathfrak{m}>\aleph_0$, but not in $\aleph_0$.

Let $\mathfrak{X}_7$ consist of axioms of groups, of the proposition

$$f_1(x,y)=f_1(y,x)$$

(commutation law) and moreover of an infinite sequence of elementary sentences α_n:

$$\prod_x \sum_y \{ \underbrace{f_1[f_1 \cdots f_1}_{(n-1)\text{ times}}(f_1(y,y),y),\ldots,y] = x$$

$$\wedge \prod_z \{\underbrace{f_1[f_1\cdots f_1}_{(n-1)\text{ times}}(f_1(z,z),z),\ldots,z] = x \to y = z\}\}.$$

In additively written groups, a_n indicates: each x is uniquely divisible by n.

Each model of $\mathfrak{X}_7$ is a group with a uniquely determined multiplication by a rational number and therefore a linear space upon the field of rational numbers. If such a linear space is of power $\mathfrak{m} > \aleph_0$ then it has a basis of power $\mathfrak{m}$, and thus two such spaces, both of power $\mathfrak{m} > \aleph_0$, are isomorphic. But there exists an infinity of different models of $\mathfrak{X}_7$ of the power $\aleph_0$. Each s-dimensional space upon the rational number forms such a model, and for different $s \leqslant \aleph_0$ those spaces, even considered only as groups, are not isomorphic.

5. Problems. Let $\mathfrak{X}$ be an elementary system; we denote by $C(\mathfrak{X})$ the set of all cardinal numbers $\mathfrak{m}$ such that $\mathfrak{X}$ is categorical in the power $\mathfrak{m}$.

The following problem arises:

P118. Which sets of cardinal numbers can be represented in the form $C(\mathfrak{X})$?

Especially

(i) Is the implication

$$\text{if } \mathfrak{n} > \mathfrak{m} \in C(\mathfrak{X}), \text{ then } \mathfrak{n} \in C(\mathfrak{X})$$

true for $\mathfrak{m} > \aleph_0$?

(ii) Is the implication

$$\text{if } \mathfrak{n} < \mathfrak{m} \in C(\mathfrak{X}) \text{ then } \mathfrak{n} \in C(\mathfrak{X})$$

true for $\mathfrak{n} > \aleph_0$?

A set C of cardinal numbers is called *definable* if there exists a (not elementary) proposition which is valid in a model $\mathfrak{M}$ if and only if $\mathfrak{M}$ is of power $\mathfrak{m} \in C$. If $\mathfrak{X}$ is a system generated by a finite set of elementary propositions (an axiomatisable system), then $C(\mathfrak{X})$ is definable.

(iii) Is it true, in general, that $C(\mathfrak{X})$ is a definable set?

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

M. T. Huber