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LENGTH, SHAPE AND AREA

BY

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This paper is an invitation for the reader to solve the problems of pure and applied geometry involved in its approach to the notion of length and shape rather than an attempt of the author to answer the questions by himself.

1. Historical background. P. S. Laplace [9] suggested as early as 1812 that probabilistic methods should be applied in measuring lengths. This challenge was answered in 1868 by M. Crofton who defined the measures of sets of straight lines in a plane. Crofton's discoveries [4] met only in part with the appreciation they deserved; his methods were applied to questions belonging to geometrical probabilities; E. Czuber [5] devotes a book to them. R. Deltheil [7] published in 1926 a monography in Borels collection; his "Probabilités géométriques" are based on the work of Elie Cartan on the principle of duality and the point of view of differential geometry is maintained throughout the whole work — in this chain no author is thoroughly conscious of his debt towards his immediate predecessor. A great part of these investigations belongs to the Integral Calculus as is shown by the title of Crofton's first paper: "... the methods used being also extended to the proof of certain new theorems in the Integral Calculus". W. Blaschke [2] and his collaborators have derived many new results from Crofton's basic idea. In Blaschke's "Integralgeometrie" we find the names of H. Lebesgue and J. Favard (1932) as the first to propose the definition of the length of an arc on Crofton's principle — a short pamphlet [12] to the same effect published by the author of this Note in 1930 seems to have escaped general notice.

2. Mathematical background. To formulate Crofton's main result let us roughly state, that having extended the notion of measure on sets of straight lines in a plane, we can speak of the probability of a line cutting an arc A in $0,1,2,\ldots$ points, the probabilities in question being proportional to the measures of the sets of lines defined respectively by the condition of their cutting A in $0,1,2\ldots$ points. Speaking the language of the theory of random variables we can designate by x the number of intersections of a random line X with A and by $E(x)$ the expected value of x: Crofton's result consists in the equality

$$cE(x) = \text{length of } A, \tag{1}$$

c being an absolute constant.

In order to formulate (1) exactly let us call Π the plane to which all our considerations will be restricted. Let O be a fixed point and OQ a fixed direction in Π. Given any angle ϑ $(0 \leqslant \vartheta < \pi)$ and any real number p, we can find a direction OT defined by $\sphericalangle QOT = \vartheta$ and the point P on OT so as to have $OP = p$. We draw through P a straight line L perpendicular to OT. Thus we have established a one-one correspondence between the points (ϑ, p) of the strip $S = \langle 0 \leqslant \vartheta < \pi)$ of the (ϑ, p)-plane and the straight lines L of Π. We can now define the measure $|Z|$ of any set Z of lines L by setting

$$|Z| = \text{plane measure of } Z^*, \tag{2}$$

where Z^* is the image of Z in S. This definition, which is a modern formulation of Crofton's basic concept, attributes a finite measure $|Z|$ to any set Z of lines L whose image Z^* has a finite plane measure in Lebesgue's sense.

Let us now consider an arc A in Π and let us call A_k $(k = 1, 2, \ldots)$ the set of lines L cutting A exactly in k points. Crofton's theorem can be written as follows:

$$\text{length of } A = \frac{1}{2} \sum_{k=1}^{\infty} k |A_k|. \tag{3}$$

To derive (1) from (3) we only have to restrict the plane Π to the circle of radius 1 and centre 0, and to assume that the arc A lies in this domain; the constant c becomes then equal to π. The restriction is obviously not essential and is needed here only to facilitate the translation of (3) into the language of probabilities.

Let us call $L(\vartheta, p)$ the image L in Π of the point (ϑ, p) in S. The number of intersections of $L(\vartheta, p)$ with an arc A in Π is a function $a(\vartheta, p)$ of two variables. The formula (3) can now be brought into the form

$$\text{length of } A = \frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{\pi} a(\vartheta, p)\, d\vartheta\, dp, \tag{4}$$

the double integral being meant in Lebesgue's sense.

The formula (4), however, is not the first to express length by a double integral: we find as early as 1832 in the works of Cauchy [3] the formula

$$\int_0^1 \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\, dt = \frac{1}{2} \int_0^{\pi} d\vartheta \int_0^1 |\dot{x}(t) \cos\vartheta + \dot{y}(t) \sin\vartheta|\, dt, \tag{5}$$

for convex curves $x = x(t)$, $y = y(t)$ $(0 \leqslant t \leqslant 1)$. It is not difficult to verify that in both formulae, (4) and (5), the double integral gives the mean length of the projections of the curve in all directions. In the light of this remark formula (4), which is a modern transcription of Crofton's theorem, appears as independent from probability considerations.

3. Proofs and generality. To verify (1) we start with the special case of A being a segment of a straight line; the proof in this case is trivial. The next step deals with an arc A composed of a finite number of straight segments $C_1, C_2, \ldots, C_n$. If x is the number of intersections of a random line X with A, and x_j $(j = 1, 2, \ldots, n)$ the number of intersections of X with C_j, then we have

$$x = \sum_{j=1}^{n} x_j,$$

and, by a theorem on random variables,

$$cE(x) = c\sum_{j=1}^{n} E(x_j) = \sum_{j=1}^{n} \text{length of } C_j = \text{length of } A.$$

The third and last step consists in approaching a given arc A by a broken line composed of consecutive chords $C_1, C_2, \ldots, C_n$; a passage to the limit for $n \to \infty$ leads eventually to (1).

For brevity's sake let us subsequently call *arc* a one-one continuous image of a finite straight segment, and a *curve* a one-one continuous image of a circle and let us call an arc or curve *rectifiable* if it has a finite length in Jordan's sense. Let us call a non-negative function $a(\vartheta, p)$ *integrable* in the strip S if it is finite in all points of S, except a set of plane L-measure zero at most, and if it becomes integrable in Lebesgue's sense over S, after changing its infinite values to finite ones — we shall assume that the double integral in (4) is computed in this manner whenever we speak of its value. These conventions enable us to state the following theorem:

(T) *For any arc A the integrability of $a(\vartheta, p)$ in S is a necessary and sufficient condition for its being rectifiable, and for any rectifiable arc A formula* (4) *holds, its left term being read as Jordan's length.*

The proof we have outlined for (1) contains already all the components of a proof of (T); the task has been done by W. Maack and published in Blaschke's book [2].

It is a natural question whether the double integral in (4) must necessarily be a Lebesgue-integral. The generality of Lebesgue-integration has two reasons:

1º it can be applied to unbounded functions, or at least to some of them,

2º it can be applied to very discontinuous functions.

It could be supposed that it is only the first property which accounts for the necessity of L-integration in (4) and in (T): there are rectifiable arcs A for which the number $a(L)$ of intersections of A with L is an unbounded function of L. Thus one is led to the question if the Riemann integral would be sufficient for arcs A for which $a(L)$ is bounded, $a(L)\leqslant 6$ for instance. H. Fast and A. Götz [8] have answered this question negatively by constructing an arc A cutting no straight line in more than 6 points and such that the function $a(\vartheta,p)$ is discontinuous in a set of positive plane measure. This example shows that it is impossible to replace the integral in (4) by an R-integral, even by changing the values of $a(\vartheta,p)$ in a set of vanishing measure.

Banach's theorem. S. Banach [1] has proved that the existence of two simple integrals

$$\int_{-\infty}^{\infty} a(\vartheta_i,p)\,dp \qquad (i=1,2;\ 0\leqslant\vartheta_1<\vartheta_2<\pi) \tag{6}$$

is a sufficient condition for A to be rectifiable. Banach's condition implies that $a(\vartheta_i,p)$ ($i=1$ or 2) is finite except for a set of p-values of linear measure 0. Banach's theorem and (T) show that functions $a(\vartheta,p)$, measurable in S and fulfilling Banach's condition must be integrable in S if they are to correspond to any arc at all. The following question arises (**P 106**): if the integrals

$$\int_0^{\pi} a(\vartheta,p)\,d\vartheta, \qquad \int_0^{\pi} a(\vartheta,-p)\,d\vartheta \tag{7}$$

are finite for $p=p_0>0$, is the double integral

$$\int_{p_0}^{\infty}\int_0^{\pi} a(\vartheta,p)\,d\vartheta\,dp \tag{8}$$

necessarily finite, supposing $a(\vartheta,p)$ corresponds to an arc A?

4. Definition of length. Dismissing the condition of A being an arc we can define the *length* of any plane set A by the formula (4) in all cases in which the double integral is finite.

The advantages of this definition are obvious: length appears here as an integral and becomes independent of the parametric representation of A, of the notion of a tangent or derivative and of the approximation of A by inscribed polygons. Our definition covers more than rectifiable arcs: a finite or an enumerable set of such arcs gets a length in the new sense if the sum of lengths of its components is finite. The field of applicability of our definition, however, is, as shown by S. Sherman [11], larger than the domain covered by the preceding statement.

The area of a surface can be defined in an analogous manner by a fourfold integral: the integrand gives the number of intersections of the surface with a variable straight line. This definition puts the paradox of H. A. Schwarz in a new light: the reason why the triangulation of a surface (of a cylinder for instance) fails in certain cases to furnish an approximation to the area is the existence of straight lines cutting the approximating triangles in more points than they cut the surface itself — the paradox disappears when such approximating polyhedra are excluded.

The only drawback to our definition of length is that it does not show immediately the length to be an invariant of rigid displacements; the proof of invariance is, however, very simple. To make our definition perfect we should consider a sphere rather than a plane.

5. Length and area on the sphere. Instead of the plane Π let us consider a sphere K of radius unity. The great circles L on K will be called *lines*. Any pair of antipodes on K will be called a *point*. For every *line* L on K we can find as its image the *point* P defined as the pair of poles coresponding to the equator L. Thus we have established a one-one correspondence $P=f(L), L=f^{-1}(P)$, between the lines L and the points P on K. We define the measure $|Z|$ of any set Z of lines L as the spherical Lebesgue-measure of the point-set $Z^*=f(Z)$, presuming the measurability of Z^*. If A is an arc on K and A_k the set of lines L cutting A in k points, Crofton's theorem can be written in the form (3), as in the case of the plane. However, we have to remember that the arc A is composed of points in the new sense adopted here — in ordinary terms it represents two antipodal arcs A', A''. If these arcs have ordinary points in common, A is not an arc in the new language because it cuts itself. When speaking of the length of A, we mean the sum of the lengths of A' and A'', each of them being computed in the ordinary way, *i. e.* after Jordan's definition. The same remark is to be applied to the measuring of Z^*.

Calling $a(P)$ the number of intersections of $L=f^{-1}(P)$ with A, we get the formula

$$\text{length of } A=\frac{1}{2}\iint_K a(P)\,dK, \tag{9}$$

the domain of integration being the sphere K and dK being the differential of area. (9) can be regarded as a definition of length for spherical arcs or, more generally, for sets of points on a sphere. This length does not change when the arc (or set) is subjected to rigid displacements on K: its invariance results trivially from the invariance of the measures A_k, any rigid displacement being a rotation of K into itself; we have only to admit that such rotations do not alter the spherical measures of point-sets.

Duality. The one-one correspondence between points and lines on K reveals a dual reciprocity of the concepts of length and area, which could hardly be discovered without spherical geometry. Let C be a convex Jordan *curve* on K; convexity means here that no L cuts C in more than two *points*, and two ordinary antipodal Jordan curves C', C'' on K with no points in common are called a *Jordan curve* on K. Calling C_k the set of *lines* L cutting C exactly in k *points* we obviously have $|C_k|=0$ for $k\neq 2$, and formula (3) gives

(10) $$\text{length of } C = |C_2|.$$

Now, the right-hand term in (10) is equal to the spherical measure of the point-set $f(C_2)$. This open set is limited by the points of $f(C_1)$; the set $f(C_1)$ is a convex Jordan *curve* on K and may be called simply $f(C)$ — this notation is compatible with the previous restriction of f to lines and sets of lines. $D=f(C)$ is a convex Jordan *curve* composed of two ordinary antipodal curves C' and C'', and the area of the annular region between C' and C'' is the spherical measure of $f(C_2)$ and thus equal to the right-hand term of (10). This enables us to write

(11) $$\text{length of } C = \text{area enclosed by } D \qquad (D=f(C)).$$

To explain the geometrical meaning of (11) we have only to realize that the symbol $D=f(C)$ signifies that the *curve* D is described by the poles of a mobile great circle touching C during its movement.

Let us state that the transformation $D=f(C)$ is an involution, which means that $D=f(C)$ implies $C=f(D)$. To verify this statement let us call L the mobile great circle touching C; the infinitesimal arcs common to C and L may be c' and c'', and I may be the diameter of L connecting c' with c''. The instantaneous movement of L is a rotation about I; the axis J, connecting the two poles of L, describes during its movement two infinitesimal arcs, parallel to c' and c''; these arcs lie on D and may be called d' and d''. The parallelity of c', c'', d', d'' and the orthogonality of I and J imply that the great circle M touching D along d' and d'' has I as the axis connecting its two poles; it obviously has J as the instantaneous axis of rotation when it moves in such a way as to remain tangent to D. It follows that the poles of M describe C during such movement, which fact is symbolized by $C=f(D)$. The involutory character of the transformation f having thus been established, we immediately get from (11) the formula

(12) $$\text{length of } D = \text{area enclosed by } C.$$

Thus we have defined an involution between annular regions on K limited by pairs of antipodal convex Jordan curves, so as to have for any two such mutually correspondent regions R_1, R_2 the dual relation

(13) $$\begin{aligned}&\text{length of boundary of } R_1 = \text{area of } R_2,\\ &\text{length of boundary of } R_2 = \text{area of } R_1.\end{aligned}$$

The involution is defined above: the poles of the great circles, touching the boundary of R_1 (R_2), describe the boundary od R_2 (R_1). L. A. Santaló [10] seems to have been the first who discovered the duality in question.

6. Practical computation of lengths. To measure the length of a river or a highway traced on a map, in conformity with formula (4), we employ a transparent sheet with a family of equidistant parallels L_i $(i=\ldots,-2,-1,0,1,2,\ldots)$. The arc A to be measured cuts L_i in a_i points, and $s_0=\sum a_i$ is the number of all intersections. Turning the sheet through an angle $\pi\cdot k/m$ $(k=0,1,\ldots,m-1)$ we get s_k intersections and the grand total is

$$N=\sum_{k=0}^{m-1} s_k.$$

Calling d the distance L_iL_{i+1}, we get the expression

(14) $$Nd\pi/2m$$

as an approximation of the length of A. The accuracy [13, 14] depends on d and m: the expression (14) tends to the length of A for $d\to 0$, $m\to\infty$ — we do not take into account the example mentioned in § 3. However, the question, whether in this case the convergence is almost sure in the stochastic sense, remains open (**P 107**). In most practical applications $d=2$ mm and $m=6$ give a sufficient accuracy. The merits of this technique appear clearly when A is a set of curves. The problem of the determination of the average declivity of a district furnishes such an example: we have to multiply the total length of the isohypses by the vertical distance of two neighbouring isohypses and to divide the product by the area of the district. It is much easier to count the intersections of the straight lines of the transparent sheet with the family of isohypses, following the straight lines, than to drive a measuring wheel along the sinuosities of the curves or to pace along them with a compass; the results obtained by these familiar methods are more liable to personal and instrumental errors than those obtained in the same time by our method.

Length of objects. A river on a map 1:100000 has in most cases a visible breadth. The geographer is interested in the length of the river, not in the length of its banks. Thus arises the concept of the length of plane objects. The method described above solves the question in a natural way: we have to count as one intersection every "bridge", *i. e.* every segment of a straight line L_i connecting one bank of the river with the opposite bank and only such segments; such procedure is equivalent to the computation of the length of the shortest submerged rope connecting the source with the mouth of the river.

7. The paradox of length. Length is a discontinuous functional. This means in plain words that we can trace in the vicinity of any rectifiable arc A another arc A' whose length exceeds an arbitrary, previously prescribed limit, or even is infinite. This fact is something more than a mathematical curiosity: it has practical consequences. When measuring the left bank of the Vistula on a school map of Poland, we get a length which is appreciably smaller than that read on a map 1:200000. When comparing the length of the present frontiers of Poland with their length in the year 963 we cannot use maps drawn with the same accuracy for both cases because of the lack of information about the precise course of our frontiers a thousand years ago. The same difficulty arises when measuring such objects as contours of leaves or perimeters of plane sections of trees: the result depends appreciably on the precision of the instruments employed.

This paradox of length is not to be confused with the fact that every measurement of physical quantities, such as areas, volumes, masses or forces, is liable to errors: when measuring an area we can fit the accuracy of instruments to the postulate of an error of less than 1%; we can, if needed, increase this accuracy to meet the demand of reducing the error beneath $1/3\%$. In most cases, it is impossible to apply such procedure to lengths. The left bank of the Vistula, when measured with increased precision would furnish lengths ten, hundred and even thousand times as great as the length read off the school map. A statement nearly adequate to reality would be to call most arcs encountered in nature not rectifiable. This statement is contrary to the belief that not rectifiable arcs are an inventions of mathematicians and that natural arcs are rectifiable: it is the opposite that is true.

Our method permits us to master the paradox of length in most practical problems: for this purpose we only have to stop the summation of series (3) at the m-th term and to write

$$\text{length of order } m \text{ of } A = \frac{1}{2}\sum_{k=1}^{m} k|A_k| \tag{15}$$

as a definition of the "length of order m". Likewise, we could modify formula (4), writing the length of order m as its left-hand term and replacing the function $a(\vartheta,p)$ in the right-hand term by $a^{(m)}(\vartheta,p)$, where $a^{(m)}=a$ for $a\leqslant m$ and $a^{(m)}=m$ for $a>m$. The computation of the length of order m by means of the transparent sheet described in § 6 is perfectly simple: we have to count the intersections of every line L_i with A; if their number a_i exceeds m we have to replace a_i by m; practically this means that we should stop counting when m is reached; this rule shortens the procedure of § 6.

This device enables us to compare the lengths of rivers or frontiers drawn with different accuracies: for instance the frontier of Poland in the year 963 is outlined in a general manner so as to have no more than 8 points in common with any straight line; the length we can obtain from such a map is exactly the length of order 8, whatever instruments we use; to compare this length with the length of the actual frontiers we must compute the length of order 8 of those frontiers, discarding in this way all those details of the modern map which would disappear if the cartographer had such scarce information about the frontiers of our time as he has about the outline they had in 10-th century [15, 6].

Relative length. The idea presented above leads to the comparison of lengths of non-rectifiable arcs. Let us write $|A|_m$ for the length of the order m of A; $|A|_m$ is always finite but $\lim_{m\to\infty}|A|_m$ is infinite if A is not rectifiable. It may happen that the arcs A and B are both not rectifiable, but the limit

$$\lim_{m\to\infty}|A|_m/|B|_m=c \tag{16}$$

is a finite number c. We may then write $|A|/|B|=c$, which means that A is c times as long as B. The device of a transparent sheet carrying parallel lines makes it possible to compute c with any desired accuracy in such practical cases as the comparison of the respective lengths of both banks of the Vistula.

8. The distance. Let A and B be two arcs (or Jordan curves) and let us designate by $b(\vartheta,p)$ the function which corresponds to B in the same way as $a(\vartheta,p)$ corresponds to A in the text of § 2. We set

$$(A,B)=\frac{1}{2}\iint_{s}|a(\vartheta,p)-b(\vartheta,p)|\,d\vartheta\,dp \tag{17}$$

as a definition of the distance (A,B) of the arcs A,B. If both arcs are rectifiable, (A,B) is finite. In this case we have $(A,B)=(B,A)$. We obviously have $(A,A)=0$. The triangular property $(A,B)+(B,C)\geqslant(A,C)$ obviously holds for rectifiable arcs. If A remains fixed and B is displaced rigidly to infinity, the distance (A,B) of the rectifiable arcs A,B becomes equal to the sum of their respective lengths. To understand properly the notion od distance as defined here we must appeal to the theorem stating that $(A,B)=0$ implies the identity of A and B. A simplified version of that theorem is this (**P108**): If A and B are arcs (or Jordan curves), $A(L)$ designates the number of intersections of A with L, and

$B(L)$ the number of intersections of B with L, then the identity of the functions $A(L) \equiv B(L)$ for L in Π implies the identity $A \equiv B$. We believe without proof this theorem to be true[1]).

The distance defined above summarizes in one number all differences between two arcs. It can be computed by means of the transparent sheet described in § 6; we only have to count the intersections along L_i with the arcs A and B: if the respective numbers are a_i and b_i, we form the sum $s_0 = \sum_i |a_i - b_i|$ and analogous sums s_k, and finally we employ formula (14). This procedure enables us to answer such questions of geography as how to determine numerically the amount of change in the course of a river.

(A,B) has the dimension of length. To make of it a pure number we have to divide it by the sum $|A|+|B|$ of the respective lengths of A and B. In this way we get an index changing from 0 to 1; the last value is assumed in the case of one of both arcs receding to infinity or shrinking to a point.

In practical problems it may be necessary to avoid the paradox of length, which affects the notion of distance as well as the notion of length itself. This is done by replacing the numbers a_i by m whenever they exceed m, and modifying b_i in the same manner — the result is a distance (A,B) of the order m, which can easily be computed by means of the transparent sheet described above.

Aproximation. If F is any family of curves and C a given curve, we can choose a curve D_0 belonging to F and minimizing the distance (C,D) in F. If, for instance, F is the family of all circles, this procedure leads to the best approximation of C by a circle. The minimum distance (C,D_0), or rather the quantity $1-(C,D_0)/(|C|+|D_0|)$, gives a measure for the "roundness" of C; the centre of the circle D_0 can be considered as the centre of C in the case of a unique solution, and the radius of D_0 may be called the radius of C.

Let any curve for which m is the maximum number of intersections with a straight line be called a *curve of the order m*. Taking for F the family of all curves of order $\leqslant m$ we may define the best approximation of C by a curve of order $\leqslant m$. This is an approach to the notion of generalization employed in cartography and in other practical questions.

Symmetry. If C is a given curve, we may define as its asymmetry relatively to a straight line M the distance (C,C') where C' designates the reflection of C in M. The line M, which minimizes the asymmetry (C,C'), may be called the *axis of symmetry* of C, and the difference

[1]) (Added in proof) It has been proved meanwhile by H. Fast; his result has been presented to the Wrocław Section of the Polish Mathematical Society the 30 October 1953.

$1-\min(C,C')/2|C|$ — the *index of symmetry* of C. We do not insist upon further details: the concepts of approximation and symmetry are not yet ready for application and are published here to give the reader an opportunity for improvements. The notion of central symmetry can be treated along the same lines. The same applies to the notion of the excentricity of a curve: if Q is any point and $C(\alpha)$ is obtained from C by turning it round Q through the angle α, we define the excentricity of C as the $\min_Q \max_\alpha (C, C(\alpha))$.

9. The components of distance. Let $\{B\}$ be the set of all arcs congruent to B and let B_0 be defined by

$$(A,B_0) = \min_{B \in \{B\}} (A,B) = c. \tag{18}$$

We set

$$(A,B) = (A,B_0) + d = c + d. \tag{19}$$

d is the component of (A,B) due to the displacement $A \to B$, whereas $c = (A,B_0)$ is the change of shape which is necessary to change A into B. The change of shape is a new sort of distance — we may call it $((A,B))$. We have $((A,B)) = 0$ if and only if A is congruent to B. We have $((A,B)) = ((B,A))$. To prove the triangular property of the change of shape let us write

$$((A,B)) = (A,B_0), \qquad ((B,C)) = (B,C_0).$$

Let us connect rigidly B with C_0 and displace both so as to bring B into the position B_0; C_0 becomes C_1 and we get

$$((A,B)) + ((B,C)) = (A,B_0) + (B,C_0) = (A,B_0) + (B_0,C_1) \geqslant (A,C_1). \tag{20}$$

Now, $((A,C))$ is the minimum of (A,X) for all X congruent to C; as C_1 is congruent to C_0 and C_0 to C, we have $(A,C_1) \geqslant ((A,C))$; comparing this with (20), we get

$$((A,B)) + ((B,C)) \geqslant ((A,C)), \qquad \text{q.e.d.}$$

We can split up the displacement into two components, the translational and the rotational one. Let $\{B\}'$ be the set of all arcs obtained from B by translations and let B_1 be defined by

$$(A,B_1) = \min_{B \in \{B\}'} (A,B) = m; \tag{21}$$

we obviously have $m \geqslant c$ and thus from (19)

$$(A,B) = (A,B_1) + d' = m + d',$$

where $d' \leqslant d$. Thus we can write

$$d = d' + r, \tag{22}$$

and call d' the translational component and r the rotational component of the displacement d. As to the change of shape we can split it up into the component due to the dilatation-concentration and the pure distortion. We shall not insist here on a detailed study of these questions.

10. Convergence. Let $\{C_n\}$ be a sequence of rectifiable arcs. The following modes of convergence deserve our attention:

1⁰ $\lim\limits_{n\to\infty} \operatorname{card} C_n = C_\infty$, which means that for almost all lines L the cardinal number of LC_n tends to the cardinal number of LC_∞;

2⁰ $\lim\limits_{n\to\infty} \operatorname{card} C_n$ exists, which means that for almost all lines L the cardinal number of LC_n tends to a limit;

3⁰ $\lim\limits_{n\to\infty} C_n = C_\infty$, which means $\lim\limits_{n\to\infty}(C_n, C_\infty) = 0$;

4⁰ $\lim\limits_{n\to\infty} C_n$ exists, which means $\lim\limits_{\substack{m\\n}\}\to\infty} (C_m, C_n) = 0$.

It is evident that 1⁰ implies 2⁰ and 3⁰ implies 4⁰. We do not know if 2⁰ implies 1⁰ and 4⁰ implies 3⁰; if it were true the question of the equivalence of 1⁰, 2⁰, 3⁰ and 4⁰ would still remain open (**P109**). It is obvious that 3⁰ implies

$$\lim_{n\to\infty} \operatorname{length} C_n = \operatorname{length} C_\infty \tag{23}$$

but we do not know if 1⁰ implies (23) (**P110**). We could define other modes of convergence based on the convergence of the sets C_nL to limiting sets, which means that for large n every point of C_nL has a small distance from the limiting set. In this manner we should get six modes of convergence 5⁰-10⁰ but we have not succeeded in establishing implications between the modes 1⁰-4⁰ and any of the six modes spoken of [2]).

There are many questions left which deserve consideration. For instance: replacing, in the integral (9), $a(P)$ by $a^2(P)$, we could define the "mean square length of A" and derive from it the "variance of A". We could also replace $|a-b|$ in (17) by $(a-b)^2$. We confine ourselves here to these vague indications and confess that we have not surmounted the difficulties encountered on the way to these and many other problems.

[2]) H. Fast has established meanwhile the equivalence of 3⁰ with (23) *plus* topological convergence $C_n \to C_\infty$. He has also found a simple example against $4^\circ \Rightarrow 3^\circ$.

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