ON MAPPINGS OF COUNTABLE SPACES

BY

M. KATĚTOV (PRAGUE)

In the present paper the following problem, due essentially to Banach[1]), is solved for a very special case, viz. for countable regular spaces: to determine the topological spaces which admit of one-to-one continuous mapping onto a compact space. It is shown that a countable regular space possesses the above property if and only if it contains no dense-in-itself set (Theorem 3). This theorem follows from Theorem 1 which asserts that such a space admits of one-to-one mapping preserving the derivatives (in G. Cantor's sense) onto a locally compact space.

All considered spaces are Hausdorff topological spaces. A space P is called *dense-in-itself* if it contains no isolated point, *dispersed* if it contains no non-void dense-in-itself set.

Let P be a space. For any ordinal number ξ, P^ξ is defined by transfinite induction as follows: $P^0=P$; if $\xi=\eta+1$, then P^ξ consists of all non-isolated points of the space P^η, and if ξ is a limit number, then $P^\xi=\prod_{\eta<\xi}P^\eta$. P^ξ is called the *ξ-th derivative* of P. Clearly, every P^ξ is closed in P.

The first and the fourth of the following lemmas are well known, the two others are easily proved by transfinite induction.

Lemma 1. A space P is dispersed if and only if $P^\xi=0$ for some ξ.

Lemma 2. If $G\subset P$ is open, then $G^\xi=GP^\xi$ for any ξ.

Lemma 3. If f is a one-to-one continuous mapping of P, then $f(P^\xi)\subset[f(P)]^\xi$ for any ξ.

Lemma 4. Every countable regular space is 0-dimensional[2]).

We introduce, for convenience, two auxiliary definitions.

A continuous mapping f of a space P into an arbitrary space is called an *α-mapping* if:

(i) f is one-to-one,

(ii) $f(P^\xi)=[f(P)]^\xi$ for any ξ,

(iii) $f(P)$ is locally compact.

A countable space P is called an *α-space* if it admits of an α-mapping, or (which is the same since every countable locally compact space is metrizable, and therefore may be imbedded [3]) into the space of real numbers) if there exists a real α-function on P

Lemma 5. Let $G_n\subset P$ ($n=1,2,\dots p$ or $n=1,2,\dots$) be closed and open, and let every G_n be an α-space. Then there exists a real α-function f on $Q=\sum_n G_n$ such that, for any compact $B\subset f(Q)$, $f^{-1}(B)\subset\sum_{k<n}G_k$ for some n.

Proof. Let $U_1=G_1$ and $U_n=G_n-\sum_{k<n}G_k$ (for $n>1$). Then U_n are disjoint; every U_n is a closed and open subset of G_n, and therefore is an α-space (cf. Lemma 2). Let g_n be a real α-function on G_n. Evidently, we may suppose g_n such that, for any $x\in G_n$, $3^{-n}\leqslant g_n(x)\leqslant 2\cdot 3^{-n}$. Now let $f(x)=g_n(x)$, if $x\in U_n$. It is easy to see that f is continuous, one-to-one, and $f(Q)$ is locally compact. If $B\subset f(Q)$, then, for some n, $y\geqslant 3^{-n}$ for every $y\in B$, whence $f^{-1}(B)\subset\sum_{k<n+1}U_k=\sum_{k<n+1}G_k$. By Lemma 2, we have, for an arbitrary ξ, $f(Q^\xi)=f(\sum_n U_n^\xi)=\sum_n f(U_n^\xi)=\sum_n[f(U_n)]^\xi=[f(Q)]^\xi$, for $f(U_n)$ are open in $f(Q)$. This proves the lemma.

Theorem 1. Let P be a countable dispersed regular space. Then there exists a bounded continuous real function f on P such that:

(i) *$f(x)=f(y)$ implies $x=y$,*

(ii) *$f(P^\xi)=[f(P)]^\xi$, for any ξ,*

(iii) *$f(P)$ is locally compact.*

Proof. It is sufficient to prove that P is an α-space. Denote by $S(\xi)$ the following proposition: if P is countable regular and $P^\xi=0$, then P is an α-space. Clearly, $S(1)$ holds so that (cf. Lemma 1) we have only to show that:

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[1]) Cf. Colloquium Mathematicum 1 (1948), p. 150, P26.

[2]) P. Urysohn, *Über die Mächtigkeit der zusammenhängenden Mengen*, Mathematische Annalen 94 (1925), p. 262-295, Kapitel II, Satz II.

[3]) Cf. C. Kuratowski, *Topologie I* (deuxième édition), Monografie Matematyczne, Warszawa-Wrocław 1948, p. 175.

1° if ξ is a limit number, and $S(\eta)$ holds for every $\eta<\xi$, then $S(\xi)$ holds,

2° $S(\xi)$ implies $S(\xi+1)$ for any ξ.

Ad 1°. Let ξ be a limit number and let $P^{\xi}=0$. If $x\epsilon P$, then, for some η, $x\epsilon P-P^{\eta}$. Since P^{η} is closed, Lemma 4 implies that there exists a closed and open G_x such that $x\epsilon G_x\subset P-P^{\eta}$. Since, by Lemma 2, $G_x^{\eta}=G_xP^{\eta}=0$, and $S(\eta)$ is supposed to hold, every G_x is an α-space and therefore, by Lemma 5, P is an α-space too.

Ad 2°. Suppose that $S(\xi)$ holds and $P^{\xi+1}=0$. First, let P^{ξ} contain exactly one point a. Lemma 4 implies that there exists, for every $x\epsilon Q=P-(a)$, an open and closed set G_x such that $x\epsilon G_x\subset Q$. By Lemma 5, there exists an α-function f on Q such that, for every compact $B\subset f(Q)$, $f^{-1}(B)\subset\sum_{x\epsilon M}G_x$, with $M\subset P$ finite, and therefore a non $\epsilon f^{-1}(B)$. Hence $f(Q)$ is not compact so that, by a well known theorem, there exists a compact space $R\supset f(Q)$ such that $R-f(Q)$ contains exactly one point b. Let $g(x)=f(x)$ for $x\epsilon Q$, and $g(a)=b$. If $B\subset R=g(P)$ is compact, and b non ϵB, then a non $\epsilon \overline{f^{-1}(B)}=g^{-1}(B)$. Hence f is continuous. By Lemma 3, $g(P^{\xi})\subset R^{\xi}$, whence $b\epsilon R^{\xi}$, and clearly $y\epsilon g(Q)$ implies y non ϵR^{ξ}. Thus $R^{\xi}=(b)$, from which we easily deduce that g is an α-mapping.

Now let P^{ξ} be arbitrary. Since P^{ξ} is closed, $(P^{\xi})^1=P^{\xi+1}=0$, and, by Lemma 4, P is 0-dimensional, there exists, for any $x\epsilon P$, a closed and open set G_x containing x and such that the set $G_x^{\xi}=G_xP^{\xi}$ contains one point at most. Then every G_x is (as we have just shown) an α-space, and therefore, by Lemma 5, P is an α-space. This completes the proof.

Lemma 6. Let P be locally compact, and let $a\epsilon P$. There exists a one-to-one continuous mapping of the space P onto a compact space T such that, for any ξ, $f(Q^{\xi})=[f(Q)]^{\xi}$, where $Q=P-(a)$. If, fore some β, P^{β} is finite and $a\epsilon P^{\beta}$, then f is an α-mapping.

Proof. Let the topology of P be modified at the point a as follows: the fundamental neighbourhoods of a are the sets $P-K$, where a non ϵK, and K is compact. We obtain a space, evidently compact, which will be denoted by T. Putting $f(x)=x$ for $x\epsilon P$, we have a one-to-one continuous mapping of P onto T. It is easy to see that $f(Q^{\xi})=[f(Q)]^{\xi}$ for any ξ. If P^{β} is finite and $a\epsilon P^{\beta}$, then $a\epsilon\overline{Q^{\xi}}$ for any $\xi<\beta$, whence $f(a)\epsilon\overline{f(Q^{\xi})}=\overline{[f(Q)]^{\xi}}$ and therefore $f(a)\epsilon T^{\beta}$. Hence f is an α-mapping.

Theorem 1 and Lemma 6 imply

Theorem 2. Let P be a countable dispersed regular space, and let, for some β, P^{β} be finite non-void. Then there exists a bounded real function f on P such that:

(i) $f(x)=f(y)$ *implies* $x=y$;

(ii) $f(P^{\xi})=[f(P)]^{\xi}$ *for any* ξ;

(iii) $f(P)$ *is compact.*

A one-to-one continuous image of a dense-in-itself set is evidently dense-in-itself. It is well-known that a countable compact space is dispersed. Thus we obtain by Theorem 1 and Lemma 6

Theorem 3. A countable regular space P admits of one-to-one mapping onto a compact space if and only if P is dispersed.
