[5] J. M. H. Olmsted, *Lebesgue theory on a Boolean algebra*, Transactions of the American Mathematical Society 51 (1942), p. 164-193.

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[8] R. Sikorski, *On the representation of Boolean algebras as fields of sets*, Fundamenta Mathematicae 35 (1948), p. 247-258.

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ON AN UNSOLVED PROBLEM FROM THE THEORY OF BOOLEAN ALGEBRAS

BY

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Let $\boldsymbol{X}$ and $\boldsymbol{I}$ be respectively a σ-field[1]) and a σ-ideal of subsets of a set $\mathscr{X}$. It is well known that the quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ is σ-complete. In some cases $\boldsymbol{X}/\boldsymbol{I}$ is further a complete[2]) Boolean algebra. The latter is true, for instance, in two following cases, where:

(a) $\boldsymbol{X}$ is a σ-field on which an enumerably additive finite measure μ is defined, and $\boldsymbol{I}$ is the σ-ideal of all sets of measure zero[3]);

(b) $\boldsymbol{X}$ is the σ-field of all subsets of a topological space[4]) $\mathscr{X}$ which possess the property of Baire[5]) (or: $\boldsymbol{X}$ is the σ-field of all Borel subsets[6]) of $\mathscr{X}$, and $\boldsymbol{I}$ is the σ-ideal of all subsets of the first category in $\mathscr{X}$.

Another kind of σ-field which is often considered beside the fields of measurable or Borel sets is the field $\boldsymbol{S}(\mathscr{X})$ of all subsets of an abstract set $\mathscr{X}$. If $\boldsymbol{I}$ is a *principal* ideal, i. e. if $\boldsymbol{I}$ is formed of all subsets of a set $X\subset\mathscr{X}$, then obviously $\boldsymbol{S}(\mathscr{X})/\boldsymbol{I}=\boldsymbol{S}(\mathscr{X}-X)$ is complete.

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[1]) Terminology and notation are in this paper the same as in my paper *The integral in a Boolean algebra*, this fascicle, p. 20-21.

[2]) A Boolean algebra $\boldsymbol{A}$ is called *complete*, if for every class $\boldsymbol{A}_0\subset\boldsymbol{A}$ there exists the least element containing all elements $A\in\boldsymbol{A}_0$.

[3]) More generally: *let $\boldsymbol{I}$ be a σ-ideal of a σ-field $\boldsymbol{X}$; if every class of disjoint sets $X\in\boldsymbol{X}-\boldsymbol{I}$ is enumerable, $\boldsymbol{X}/\boldsymbol{I}$ is complete.*

[4]) A space is called *topological*, if it fulfils the four well-known axioms of Kuratowski. See e. g. P. Alexandroff und H. Hopf, *Topologie*, Berlin, 1935, p. 37.

[5]) A subset X of a topological space possesses the *property of Baire* if it can be represented in the form $X=G+P-R$, where G is open, and P and R are of the first category. See C. Kuratowski, *Topologie I*, Monografie Matematyczne, Warszawa-Lwów 1933, p. 49.

[6]) If $\boldsymbol{X}_1$ is the field of all subsets with the property of Baire, $\boldsymbol{X}_2$ is the field of Borel sets, and $\boldsymbol{I}$ is the ideal of sets of the first category, then the algebras $\boldsymbol{X}_1/\boldsymbol{I}$ and $\boldsymbol{X}_2/\boldsymbol{I}$ are isomorphic.

The question arises whether principal ideals are the only σ-ideals $\boldsymbol{I}$ of $\boldsymbol{S}(\mathscr{X})$, such that $\boldsymbol{S}(\mathscr{X})/\boldsymbol{I}$ is a complete Boolean algebra. On account of (a) the answer is negative, if there exists an enumerably additive finite measure μ defined for all subsets of $\mathscr{X}$ and vanishing for all one-point sets ($\mu(\mathscr{X})>0$). Banach and Ulam have proved [7]) that such a measure μ does not exist if the potency of $\mathscr{X}$ is less than the first aleph inaccessible in the weak sense [8]). The hypothesis that $2^{\aleph_0}$ is less than the first aleph inaccessible in the weak sense implies that such a measure μ does not exist also if the potency of $\mathscr{X}$ is less than the first aleph inaccessible in the strict sense [9]). Thus the problem may be formulated in the following way:

P61. *$\mathscr{X}$ is a set of potency less than the first aleph inaccessible in the weak (strict) sense. Are principal ideals the only σ-ideals $\boldsymbol{I}$ of $\boldsymbol{S}(\mathscr{X})$ such that $\boldsymbol{S}(\mathscr{X})/\boldsymbol{I}$ is complete?* [10]).

On account of (a) the positive answer to this problem would imply the above mentioned theorems of Banach and Ulam on the non-existence of non-trivial measures on $\boldsymbol{S}(\mathscr{X})$; by (b) this answer would imply also that, if a topological space $\mathscr{X}$ (of potency less than the first inaccesible aleph) dense in itself is not of the first category in itself, it contains a subset which does not possess the property of Baire.

I shall prove the following theorem:

Let $\boldsymbol{I}$ be an ideal (of subsets of a set $\mathscr{X}$) containing all one-point sets, and let $\mathfrak{m}(\boldsymbol{I})$ be the least cardinal $\mathfrak{m}$ with the property: there exists a class $\boldsymbol{X}_0\subset\boldsymbol{S}(\mathscr{X})$ of potency $\mathfrak{m}$, such that $\boldsymbol{I}$ is formed of all subsets of sets belonging to $\boldsymbol{X}_0$. Then, if there exists a class $\boldsymbol{X}_1$ of potency $\mathfrak{m}(\boldsymbol{I})$ of disjoint sets which do not belong to $\boldsymbol{I}$, the Boolean algebra $\boldsymbol{S}(\mathscr{X})/\boldsymbol{I}$ is not complete [11]).

In order to prove this theorem it is sufficient to show that if $X-X_1\epsilon\boldsymbol{I}$ for every $X\epsilon\boldsymbol{X}_1$, then there exists a set $X_2\subset X_1$ such that $X-X_2\epsilon\boldsymbol{I}$ for every $X\epsilon\boldsymbol{X}_1$, and X_1-X_2 non $\epsilon\boldsymbol{I}$.

Let φ be a one-one mapping of the class $\boldsymbol{X}_1$ in the class $\boldsymbol{X}_0$. Since $\varphi(X)\epsilon\boldsymbol{I}$ and $X-X_1$ non $\epsilon\boldsymbol{I}$ for every $X\epsilon\boldsymbol{X}_1$, we have $X-X_1-\varphi(X)\neq 0$. Let $x(X)$ be an element of the set $X-X_1-\varphi(X)$, let X_0 denote the set of all elements $x(X)$, where $X\epsilon X_1$, and let $X_2=X_1-X_0$. Since $x(X)$ non $\epsilon\varphi(X)$, the set X_0 is not a subset of a set belonging to $\boldsymbol{X}_0$; hence X_0 non $\epsilon\boldsymbol{I}$. Since $x(X)\epsilon\boldsymbol{I}$, we have $X-X_2=X-X_1+x(X)\epsilon\boldsymbol{I}$ for every $X\epsilon X_1$, q. e. d.

On the other hand, there exist σ-ideals $\boldsymbol{I}$, such that $\mathfrak{m}(\boldsymbol{I})>\overline{\overline{\mathscr{X}}}$; such ideals do not fulfil the assumptions of the theorem just proved. For instance, let $\mathscr{X}$ be a set of potency $2^{\aleph_0}$ and let $\boldsymbol{X}^0$ be a class of enumerably independent sets [12]) with $\overline{\overline{\boldsymbol{X}}}_0=2^{2^{\aleph_0}}$. The class $\boldsymbol{I}$ of all subsets of all enumerable sums of sets $X\epsilon\boldsymbol{X}^0$ is an σ-ideal, such that $\mathfrak{m}(\boldsymbol{I})=2^{2^{\aleph_0}}>2^{\aleph_0}$.

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[7]) S. Banach, *Über additive Massfunktionen in abstrakten Mengen*, Fundamenta Mathematicae 15 (1930), p. 97-101; S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fundamenta Mathematicae 16 (1930), p. 140-150.

[8]) $\mathfrak{p}=\aleph_\lambda>\aleph_0$ is called *inaccesible in the weak sense*, if λ is a limit number and if the condition $\mathfrak{p}_t<\mathfrak{p}$, where t runs over a set T of potency less than $\mathfrak{p}$, implies $\sum_{t\epsilon T}\mathfrak{p}_t<\mathfrak{p}$.

[9]) A cardinal $\mathfrak{p}>\aleph_0$ is called *inaccessible in the strict sense*, if it is inaccessible in the weak sense and if, moreover, $\mathfrak{m}^{\mathfrak{n}}<\mathfrak{p}$ for every $\mathfrak{m}<\mathfrak{p}$ and $\mathfrak{n}<\mathfrak{p}$. See A. Tarski, *Über unerreichbare Kardinalzahlen*, Fundamenta Mathematicae 30 (1938), p. 69.

[10]) This problem is unsolved also in the case where the space is of potency $\aleph_1$ or $2^{\aleph_1}$.

[11]) Let R denote the set of all real numbers, and let $\boldsymbol{M}$ and $\boldsymbol{N}$ denote respectively: the σ-ideal of all sets $X\subset R$ of measure zero, and the σ-ideal of all sets $X\subset R$ of the first category. By the theorem above the Boolean algebra $\boldsymbol{S}(R)/\boldsymbol{M}$ is not complete. Similarly, the hypothesis $2^{\aleph_0}=\aleph_1$ implies that the Boolean algebra $\boldsymbol{S}(R)/\boldsymbol{N}$ is not complete. See W. Sierpiński, *Hypothèse du continu*, Monografie Matematyczne, Warszawa-Lwów, 1934, p. 109-110.

[12]) See E. Marczewski, *Ensembles indépendants et leurs applications à la théorie de la mesure*, Fundamenta Mathematicae 35 (1948), p. 21-22.