THE INTEGRAL IN A BOOLEAN ALGEBRA

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In this paper I give a definition of Lebesgue’s integral in a σ-complete Boolean algebra.

The main difficulty in the generalization of the theory of the integral in an abstract space to the case of a Boolean algebra A lies in the necessity of replacing the notion of real point function by another notion, which can be expressed in terms of the theory of Boolean algebras. A generalization of the notion of real point function I consider in this paper is additive homomorphisms mapping the field C of all Borel sets of real numbers in A. The basis of the definition of the integral is that every σ-complete Boolean algebra may be considered as a quotient algebra X/I where X is a σ-complete field of subsets of a set X, and I is a σ-ideal, and that every homomorphism f of C in A is induced by a real function f of a point of X. The integral of a homomorphism f of a set X/I in A is then defined as the integral of the function f which induces f.

It is characteristic for this kind of definition of the integral that all the properties of the integral in a σ-complete Boolean algebra are immediate consequences of those of the integral in an abstract space X. This definition shows also that the generalization of the theory of the integral in an abstract space to the case of a Boolean algebra is in fact not essential, since the examination of the integral in a Boolean algebra A can be always reduced to the examination of the integral in an abstract space X.

Terminology and notation. A will always denote a σ-complete Boolean algebra. Elements of A will be denoted by A, A, A, ...

The sum (joint) of a finite or enumerable sequence {Aj} of elements of A will be denoted by A + A + ... or by ∑ Aj.

A will denote the complement of the element A of A; A, A, A will denote the product (meet) of the elements A, A, A. |A| will denote the unit of A, i.e., an element such that A|A| = A.

If A, A, A = A', we say that A, A, A are disjoint.

A real non-negative function μ(A) of a quotient algebra A is called a measure on A. B will always denote the field of all Borel subsets of real numbers (+∞ and −∞ being considered also as real numbers).

Elements of B will be denoted by B, B, B, ...

A mapping f of B in A will be called a homomorphism if f(B') = f(B') for each B in A and if f(Σ B) = Σ f(B) for each enumerable sequence {B} of Borel sets.

Let X be an arbitrary set and X a σ-field of subsets of X. A set X is measurable on X if φ(φ(B)) is a homomorphism of B in X. i.e., φ(φ(B)) = φ(B) for every B in C.

A quotient algebra X/I is called a σ-quotient algebra of the set X if X is a σ-field of subsets of X and I is a σ-ideal of subsets of X. Elements of X/I are disjoint classes of sets X ∈ X such that two sets X, X belong to the same class if and only if (X, X) − (X, X) intersection of X with X contains an element of X, which will be denoted by (X).

σ-quotient algebras are obviously σ-complete Boolean algebras. Conversely, every σ-complete Boolean algebra is isomorphic to a σ-quotient algebra.

I. Let A be a σ-complete Boolean algebra. Consider a σ-quotient algebra X/I (of a set X), which is isomorphic to A. Let h be an isomorphism of X/I on A. It is known that for each ho
nomorphism \( f \) of \( B \) in \( A \) there exists a measurable \( (X) \) function \( \varphi \) defined on \( \mathcal{B} \) such that
\[
 f(B) = h(\varphi^{-1}(B)) \quad \text{for each } B \in \mathcal{B}.
\]
We shall say that the function \( \varphi \) induces the homomorphism \( f \).

Conversely, every measurable \( (X) \) function \( \varphi \) defined on \( \mathcal{B} \) induces a homomorphism \( f \) of \( B \) in \( A \) defined by the formula (i). Two measurable \( (X) \) functions \( \varphi_1 \) and \( \varphi_2 \) induce the same homomorphism \( f \) if and only if
\[
 \sum_{x} (\varphi_1(x) - \varphi_2(x)) \in \mathbb{I}.
\]

The above mentioned correspondence between homomorphisms of \( B \) in \( A \) and measurable \( (X) \) functions permits us easily to define the algebraic operations on homomorphisms. Let \( f_t \) and \( f_{\bar{t}} \) be two homomorphisms induced respectively by measurable \( (X) \) functions \( \varphi_t \) and \( \varphi_{\bar{t}} \). We define the sum \( f_t + f_{\bar{t}} \), the difference \( f_t - f_{\bar{t}} \), the product \( f_t \cdot f_{\bar{t}} \) and the quotient \( f_t/f_{\bar{t}} \) as homomorphisms induced by the measurable \( (X) \) functions \( \varphi_t + \varphi_{\bar{t}}, \varphi_t - \varphi_{\bar{t}}, \varphi_t \cdot \varphi_{\bar{t}} \) and \( \varphi_t/\varphi_{\bar{t}} \) respectively. It is easy to show that the homomorphisms \( f_t + f_{\bar{t}}, f_t - f_{\bar{t}}, f_t \cdot f_{\bar{t}} \) and \( f_t/f_{\bar{t}} \) do not depend on the choice of the inducing functions \( \varphi_t \) and \( \varphi_{\bar{t}} \). They do not either depend on the choice of the \( \sigma \)-quotient algebra \( X/I \) isomorphic to \( A \).

We shall prove this fact only for the sum \( f_t + f_{\bar{t}} \). The remaining cases can be proved in an analogous way. Let \( B(r) \) denote the set of all real numbers greater than a rational number \( r \). It is easy to show that for \( r > 0 \)
\[
f(B(r)) = \sum f_t(B(r') \cdot f_{\bar{t}}(B(r - r'))
\]
and for \( r < 0 \)
\[
f(B(r)) = \sum f_t(B(r') \cdot f_{\bar{t}}(B(r - r')) + f_t([-\infty, 0)) \cdot f_{\bar{t}}([0, \infty)) + f_t([0, \infty)) \cdot f_{\bar{t}}([-\infty, 0))
\]
(\( r' \) denotes here an arbitrary rational number). Hence \( f(B(r)) \) is independent of the choice of the \( \sigma \)-quotient algebra \( X/I \). On the other hand, if \( \varphi_t \) and \( \varphi_{\bar{t}} \) are two homomorphisms of \( B \) in \( A \), and \( \varphi_t(B(r)) = \varphi_{\bar{t}}(B(r)) \) for every rational number \( r \), then \( h_t = h_{\bar{t}} \).

\(^7\) See Sikorski (9), theorem 4.5.
\(^8\) We define the elementary algebraic operations on infinite real numbers as in the book of Saks (7), p. 6.
\(^9\) For we assumed in the footnote \(^7\) that \( [+\infty) + [-\infty, 0) = 0 \).

Therefore the sum \( f_t + f_{\bar{t}} \) is independent of the choice of \( X/I \). Analogously we say that a homomorphism \( f \) of \( B \) in \( A \) is
(a) simple, (b) non-negative, if it is induced by a measurable \( (X) \) function \( \varphi \) which is (a) simple, (b) non-negative respectively. This definition does not depend on \( X/I \) since if \( f \) is (a) simple, (b) non-negative, if and only if respectively (a) there exists a finite set \( \mathcal{B} = \{r_1, r_2, \ldots, r_n\} \) such that \( f(B) = |A| \); (b) \( f(B) = |A| \), where \( B \) denotes the set of all real non-negative numbers.

If \( f \) is a homomorphism of \( B \) in \( A \) induced by a function \( \varphi \), \( f \) and \( f \) will denote homomorphisms induced respectively by
\[
\max \, (\varphi(x), 0) \quad \text{and} \quad \max \, (-\varphi(x), 0)
\]
The homomorphisms \( f \) and \( f \) are non-negative, they do not depend on the isomorphic \( \sigma \)-quotient algebra \( X/I \), and \( f = f - f \).

Let now \( [f_t] \) be an enumerable sequence of homomorphisms.

We say that a homomorphism \( f_k \) is the limit of the sequence \( f_n \) if there exist functions \( \varphi_n \) \((n = 0, 1, 2, \ldots)\) inducing \( f_n \) such that \( \varphi_n = \lim \varphi_n \).

The method of generalization of other definitions from the theory of the integral is clear.

II. Let \( \mu \) be a measure on a given \( \sigma \)-complete Boolean algebra \( A \), and let \( X/I \) and \( h \) have the same meaning as before. The function
\[
\bar{\mu}(X) = \mu(h([X/I]))
\]
is a measure on \( X \). In particular, if \( X = X/I \), then \( \mu(X) = 0 \), and consequently, by (ii), if two measurable \( (X) \) functions \( \varphi \) and \( \varphi_2 \) induce the same homomorphism \( f \), then
\[
\bar{\mu}(\sum (\varphi_1(x) \neq \varphi_2(x))) = 0.
\]

Let \( f \) be a homomorphism of \( B \) in \( A \), and let \( A \subseteq A \). Let \( \varphi \) be a measurable \( (X) \) real function which induces \( f \), and let \( X \) be an
\(^{10\text{a}}\) A function \( \varphi \) is simple if \( \varphi(X) \) is a finite set of finite real numbers. See Saks (7), p. 7.
\(^{10\text{b}}\) See ibidem, p. 15.
element of $X$, such that $A = [X]$. We say that the homomorphism $f$ possesses a definite integral $(A, \mu)$ over $A$ if the function $\varphi$ possesses a definite integral\(^{19}\) $(X, \mu)$ over $X$.

(iv) 

\[
(\int_X \varphi \, d\mu) = \mu(\varphi).
\]

The integral (iv) is independent of the choice of a representative $X$ of $A$ since $A = [X] = [X']$ implies $\mu(X - X') + (X' - X) = 0$. By (iii), the integral (iv) is also independent of the inducing function $\varphi$. The number (iv) will be called the definite integral $(A, \mu)$ of the homomorphism $f$ over $A$ and will be denoted by

(v) 

\[
(A) \int f \, d\mu.
\]

It follows immediately from the above mentioned definition and from the consideration of part (i) that the integral (v) so defined in a $\sigma$-complete Boolean algebra possesses all the properties of the integral in an abstract space. All theorems on the integral proved in Chapter I of Sak's [1] are true also in the case of the integral in a $\sigma$-complete Boolean algebra. We must only modify accordingly several definitions from the theory of the integral in an abstract space\(^{20}\).

In particular, if a homomorphism $f$ possesses a definite integral $(A, \mu)$ over $A \subset A$, then

(vi) 

\[
(A) \int f \, d\mu = (A) \int f \, d\mu - (A) \int f \, d\mu.
\]

There exists a non-decreasing\(^{19}\) sequence $\{f_n\}$ of simple non-negative homomorphisms, such that $f = \lim f_n$. If $R_n = [r_1, r_2, \ldots, r_m]$ is a set of finite real numbers, such that $f_n(R_n) = A$, we have

\[
(A) \int f_n \, d\mu = \sum_{i=1}^{m} \mu(A - f(r_i)) = r_i.
\]

Hence $(A) \int f_n \, d\mu$ does not depend on the choice of the iso-

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\(^{19}\) The terminology and notation from the theory of the integral in an abstract space are in this paper the same as in Saks [7].

\(^{20}\) For instance, instead of additive functions of a set we must consider additive functions of an element of $A$, instead of measurable functions we must consider always homomorphisms of $B$ in $A$, etc.

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morphic $\sigma$-quotient algebra $X/I$. By Lebesgue’s theorem on the integration of monotone sequences\(^{17}\), we find that the integral

\[
(A) \int f \, d\mu = \lim (A) \int f_n \, d\mu
\]
does not depend on $X/I$. Analogously we infer that $(A) \int f \, d\mu$, and consequently, by (vi), also the integral (v), do not depend on the isomorphic algebra $X/I$.

We must still prove that in the case where the Boolean algebra under consideration is a $\sigma$-field $X$ of subsets of an abstract space $\mathcal{X}$, the definition (v) coincides with the usual definition of the integral in the abstract space $\mathcal{X}$. As a $\sigma$-quotient algebra which is isomorphic to $X$, we may select the algebra $X/0$ where $0$ denotes the ideal containing one element only: the empty set. Then $\{X\} = (X)$ and $\mu(X) = \mu(X)$ for each $X \in X$. By condition (ii) we obtain that the equation

(vii) 

\[
\int f \, d\mu = \int f \, d\mu
\]

establishes a one-one correspondence between the class of all homomorphisms $f$ in $B$ and the class of all measurable $(X)$ functions. By definition (v)

\[
(X) \int f \, d\mu = (X) \int f \, d\mu,
\]

where $f = \varphi^{-1}$. With respect to the correspondence (vii) the definition (v) is in fact a generalization of the definition of the integral in an abstract space.

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REFERENCES


ON AN UNSOLVED PROBLEM
FROM THE THEORY OF BOOLEAN ALGEBRAS

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Let $X$ and $I$ be respectively a $\sigma$-field $^1$ and a $\sigma$-ideal of subsets of a set $\mathcal{X}$. It is well known that the quotient algebra $X/I$ is $\sigma$-complete. In some cases $X/I$ is further a complete $^2$ Boolean algebra. The latter is true, for instance, in two following cases, where:

(a) $X$ is a $\sigma$-field on which an enumerably additive finite measure $\mu$ is defined, and $I$ is the $\sigma$-ideal of all sets of measure zero $^3$;

(b) $X$ is the $\sigma$-field of all subsets of a topological space $\mathcal{X}$ $^4$ which possess the property of Baire $^5$ (or: $X$ is the $\sigma$-field of all Borel subsets $^6$ of $\mathcal{X}$, and $I$ is the $\sigma$-ideal of all subsets of the first category in $\mathcal{X}$).

Another kind of $\sigma$-field which is often considered besides the fields of measurable or Borel sets is the field $S(\mathcal{X})$ of all subsets of an abstract set $\mathcal{X}$. If $I$ is a principal ideal, i.e., if $I$ is formed of all subsets of a set $X \subset \mathcal{X}$, then obviously $S(\mathcal{X})/I = S(\mathcal{X} - X)$ is complete.

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$^1$ Terminology and notation are in this paper the same as in my paper "The integral in a Boolean algebra," this fascicle, p. 20-21.

$^2$ A Boolean algebra $A$ is called complete, if for every class $A \subseteq A$ there exists the least element containing all elements $A \vee A'$.

$^3$ More generally: let $A$ be a $\sigma$-ideal of a $\sigma$-field $X$; if every class of disjoint sets $X \lambda X - I$ is enumerable, $X/I$ is complete.

$^4$ A space is called topological, if it fulfills the four well-known axioms of Kuratowski. See e.g., P. Alexandroff und H. Hopf, Topologie, Berlin, 1935, p. 37.

$^5$ A subset $X$ of a topological space possesses the property of Baire if it can be represented in the form $X = G + P - R$, where $G$ is open, and $P$ and $R$ are of the first category. See C. Kuratowski, Topologie I, Monografie Matematyczne, Warszawa-Lwów 1933, p. 49.

$^6$ If $X$ is the field of all subsets with the property of Baire, $X$ is the field of Borel sets, and $I$ is the ideal of sets of the first category, then the algebras $X/I$ and $X/I$ are isomorphic.