THE INTEGRAL IN A BOOLEAN ALGEBRA

BY

R. SIKORSKI (WARSAW)

In this paper I give a definition of Lebesgue's integral in a σ-complete Boolean algebra[1].

The main difficulty in the generalization of the theory of the integral in an abstract space to the case of a Boolean algebra $\boldsymbol{A}$ lies in the necessity of replacing the notion of real point function by another notion, which can be expressed in terms of the theory of Boolean algebras. As a generalization of the notion of real point function I consider in this paper $\aleph_0$-additive homomorphisms mapping the field $\boldsymbol{B}$ of all Borel sets of real numbers in $\boldsymbol{A}$[2]. The basis of the definition of the integral is that every σ-complete Boolean algebra $\boldsymbol{A}$ may be considered as a quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ (where $\boldsymbol{X}$ is a σ-complete field of subsets of a set $\mathscr{X}$ and $\boldsymbol{I}$ is a σ-ideal), and that every homomorphism f of $\boldsymbol{B}$ in $\boldsymbol{A}$ is induced[3] by a real function φ of a point of $\mathscr{X}$. The integral of a homomorphism f of $\boldsymbol{B}$ in $\boldsymbol{A}$ is then defined as the integral of the function φ which induces f.

It is characteristic for this kind of definition of the integral that all the properties of the integral in a σ-complete Boolean algebra are immediate consequences of those of the integral in an abstract space $\mathscr{X}$. This definition shows also that the generalization of the theory of the integral in an abstract space to the case of a Boolean algebra is in fact not essential, since the examination of the integral in a Boolean algebra $\boldsymbol{A}$ can be always reduced to the examination of the integral in an abstract space $\mathscr{X}$.

Terminology and notation. $\boldsymbol{A}$ will always denote a *σ-complete Boolean algebra*[4]. Elements of $\boldsymbol{A}$ will be denoted by $A, A_1, A_2, \dots$

[1] This subject has already been considered by other writers. See e.g. Carathéodory [3], Olmsted [5], Bischof [2], and Ridder [7].

[2] The idea of the application of $\aleph_0$-additive homomorphisms of $\boldsymbol{B}$ in $\boldsymbol{A}$ (as a generalization of the notion of a real point function in the theory of the integral) is due to Marczewski.

[3] See the definition, p. 22.

[4] For σ-complete Boolean algebras (or Boolean σ-algebras) see Birkhoff [1], p. 1, 29 and 88.

The *sum* (joint) of a finite or enumerable sequence $\{A_n\}$ of elements of $\boldsymbol{A}$ will be denoted by $A_1+A_2+\dots$ or by $\sum_n A_n$.

A' will denote the *complement* of the element $A \epsilon \boldsymbol{A}$; $A_1 \cdot A_2$ will denote the *product* (meet) of the elements A_1 and A_2.

$|\boldsymbol{A}|$ will denote the *unit* of $\boldsymbol{A}$, i.e. an element such that $A \cdot |\boldsymbol{A}| = A$.

If $A_1 \cdot A_2 = |\boldsymbol{A}|'$, we say that A_1 and A_2 are disjoint.

A real non-negative function $\mu(A)$ of element $A \epsilon \boldsymbol{A}$, such that $\mu(\sum_n A_n) = \sum_n \mu(A_n)$ for each enumerable sequence $\{A_n\}$ of disjoint elements of $\boldsymbol{A}$, is called a *measure* on $\boldsymbol{A}$.

$\boldsymbol{B}$ will always denote the field of all Borel subsets of real numbers ($+\infty$ and $-\infty$ being considered also as real numbers). Elements of $\boldsymbol{B}$ will be denoted by $B, B_1, B_2, \dots$

A mapping f of $\boldsymbol{B}$ in $\boldsymbol{A}$ will be called a *homomorphism* if $f(B') = (f(B))'$ for each $B \epsilon \boldsymbol{B}$ and if $f(\sum_n B_n) = \sum_n f(B_n)$ for each enumerable sequence $\{B_n\}$ of Borel sets.

Let $\boldsymbol{X}$ be a *σ-field*[5] of subsets of a set $\mathscr{X}$. A real function defined on $\mathscr{X}$ is called *measurable* ($\boldsymbol{X}$) if $\varphi^{-1}(B)$ is a homomorphism of $\boldsymbol{B}$ in $\boldsymbol{X}$, i. e. $\varphi^{-1}(B) \epsilon \boldsymbol{X}$ for every $B \epsilon \boldsymbol{B}$.

A quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ is called a *σ-quotient algebra* (of the set $\mathscr{X}$) if $\boldsymbol{X}$ is a σ-field of subsets of $\mathscr{X}$ and $\boldsymbol{I}$ is a σ-ideal[6] of subsets of $\mathscr{X}$. Elements of $\boldsymbol{X}/\boldsymbol{I}$ are disjoint classes of sets $X \epsilon \boldsymbol{X}$ such that two sets X_1, X_2 belong to the same class if and only if $(X_1 - X_2) + (X_2 - X_1) \epsilon \boldsymbol{I}$. The element of $\boldsymbol{X}/\boldsymbol{I}$ containing an $X \epsilon \boldsymbol{X}$ will be denoted by $[X]$.

σ-quotient algebras are obviously σ-complete Boolean algebras. Conversely, every σ-complete Boolean algebra is isomorphic to a σ-quotient algebra[7].

I. Let $\boldsymbol{A}$ be a σ-complete Boolean algebra. Consider a σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ (of a set $\mathscr{X}$), which is isomorphic to $\boldsymbol{A}$. Let h be an isomorphism of $\boldsymbol{X}/\boldsymbol{I}$ on $\boldsymbol{A}$. It is known[8] that for each ho-

[5] i. e. a complementative and enumerably additive class of sets.

[6] i. e. $\boldsymbol{I}$ is a class of subsets of $\mathscr{X}$ such that $X \epsilon \boldsymbol{I}$ and $X_0 \subset X$ imply $X_0 \epsilon \boldsymbol{I}$, and $X_n \epsilon \boldsymbol{I}$ $(n = 1, 2, \dots)$ implies $X_1 + X_2 + \dots \epsilon \boldsymbol{I}$.

[7] See Loomis [4], p. 757 and Sikorski [8], theorem 5.3.

[8] See Sikorski [9], theorem 3.1.

momorphism f of $\boldsymbol{B}$ in $\boldsymbol{A}$ there exists a measurable $(\boldsymbol{X})$ function φ defined on $\mathcal{X}$ such that

(i) $$f(B)=h([\varphi^{-1}(B)]) \quad \text{for each } B\in\boldsymbol{B}.$$

We shall say that the function φ *induces* the homomorphism f.

Conversely, every measurable $(\boldsymbol{X})$ function φ defined on $\mathcal{X}$ induces a homomorphism f of $\boldsymbol{B}$ in $\boldsymbol{A}$ defined by the formula (i). Two measurable $(\boldsymbol{X})$ functions φ_1 and φ_2 induce the same homomorphism f if and only if [9])

(ii) $$\sum_x(\varphi_1(x)\neq\varphi_2(x))\in\boldsymbol{I}.$$

The above mentioned correspondence between homomorphisms of $\boldsymbol{B}$ in $\boldsymbol{A}$ and measurable $(\boldsymbol{X})$ functions permits us easily to define the algebraic operations on homomorphisms. Let f_1 and f_2 be two homomorphisms induced respectively by measurable $(\boldsymbol{X})$ functions φ_1 and φ_2. We define the sum f_1+f_2, the difference f_1-f_2, the product $f_1\cdot f_2$ and the quotient f_1/f_2 as homomorphisms induced by the measurable $(\boldsymbol{X})$ functions $\varphi_1+\varphi_2$, $\varphi_1-\varphi_2$, $\varphi_1\cdot\varphi_2$ and φ_1/φ_2 respectively [10]). It is easy to show that the homomorphisms f_1+f_2, f_1-f_2, $f_1\cdot f_2$ and f_1/f_2 do not depend on the choice of the inducing functions φ_1 and φ_2. They do not either depend on the choice of the σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ isomorphic to $\boldsymbol{A}$.

We shall prove this fact only for the sum $f=f_1+f_2$ (the remaining cases can be proved in an analogous way). Let $B(r)$ denote the set of all real numbers greater than a rational number r. It is easy to show that for $r\geqslant 0$

$$f(B(r))=\sum_{r'}f_1(B(r'))\cdot f_2(B(r-r'))$$

and for $r<0$ [11])

$$f(B(r))=\sum_{r'}f_1(B(r'))\cdot f_2(B(r-r'))+f_1((+\infty))\cdot f_2((-\infty))+f_1((-\infty))\cdot f_2((+\infty))$$

(r' denotes here an arbitrary rational number). Hence $f(B(r))$ is independent of the choice of the σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$. On the other hand, if g_1 and g_2 are two homomorphisms of $\boldsymbol{B}$ in $\boldsymbol{A}$, and $g_1(B(r))=g_2(B(r))$ for every rational number r, then $g_1=g_2$.

---

[9]) See Sikorski [9], theorem 4.5.

[10]) We define the elementary algebraic operations on infinite real numbers as in the book of Saks [7], p. 6.

[11]) For we assumed in the footnote 10 that $(+\infty)+(-\infty)=0$.

Therefore the sum f_1+f_2 is independent of the choice of $\boldsymbol{X}/\boldsymbol{I}$.

Analogously we say that a homomorphism f of $\boldsymbol{B}$ in $\boldsymbol{A}$ is (a) simple, (b) non-negative, if it is induced by a measurable $(\boldsymbol{X})$ function φ which is (a) simple [12]), (b) non-negative respectively. This definition does not depend on $\boldsymbol{X}/\boldsymbol{I}$ since f is (a) simple, (b) non-negative, if and only if respectively (a) there exists a finite set $R=(r_1,r_2,\ldots r_n)$ such that $f(R)=|\boldsymbol{A}|$; (b) $f(B_0)=|\boldsymbol{A}|$, where B_0 denotes the set of all real non-negative numbers.

If f is a homomorphism of $\boldsymbol{B}$ in $\boldsymbol{A}$ induced by a function φ, $\overset{\circ}{f}$ and $\underset{\circ}{f}$ will denote homomorphisms induced respectively by $\max(\varphi(x),0)$ and $\max(-\varphi(x),0)$ [13]). The homomorphisms $\overset{\circ}{f}$ and $\underset{\circ}{f}$ are non-negative, they do not depend on the isomorphic σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$, and $f=\overset{\circ}{f}-\underset{\circ}{f}$.

Let now $\{f_n\}$ be an enumerable sequence of homomorphisms.

We say that a homomorphism f_0 is the *limit* of the sequence f_n if there exist functions φ_n $(n=0,1,2,\ldots)$ inducing f_n such that $\varphi_0=\lim_n\varphi_n$.

The method of generalization of other definitions from the theory of the integral is clear.

II. Let μ be a measure on a given σ-complete Boolean algebra $\boldsymbol{A}$, and let $\boldsymbol{X}/\boldsymbol{I}$ and h have the same meaning as before. The function

$$\bar{\mu}(X)=\mu(h([X])) \qquad \text{for } X\in\boldsymbol{X}$$

is a measure on $\boldsymbol{X}$. In particular, if $X\in\boldsymbol{X}\cdot\boldsymbol{I}$, then $\bar{\mu}(X)=0$, and consequently, by (ii), if two measurable $(\boldsymbol{X})$ functions φ_1 and φ_2 induce the same homomorphism f, then

(iii) $$\bar{\mu}\Big(\sum_x(\varphi_1(x)\neq\varphi_2(x))\Big)=0.$$

Let f be a homomorphism of $\boldsymbol{B}$ in $\boldsymbol{A}$, and let $A\in\boldsymbol{A}$. Let φ be a measurable $(\boldsymbol{X})$ real function which induces f, and let X be an

---

[12]) A function φ is *simple* if $\varphi(\boldsymbol{X})$ is a finite set of finite real numbers. See Saks [7], p. 7.

[13]) See ibidem, p. 13.

element of $\boldsymbol{X}$, such that $A=[X]$. We say that the homomorphism f *possesses a definite integral* $(\boldsymbol{A},\mu)$ over A if the function φ possesses a definite integral[14] $(\boldsymbol{X},\bar{\mu})$ over $\boldsymbol{X}$

(iv) $$(\boldsymbol{X})\int_X \varphi\, d\bar{\mu}.$$

The integral (iv) is independent of the choice of a representative X of A since $A=[X]=[X_1]$ implies $\bar{\mu}[(X-X_1)+(X_1-X)]=0$. By (iii), the integral (iv) is also independent of the inducing function φ. The number (iv) will be called the *definite integral* $(\boldsymbol{A},\mu)$ of the homomorphism f over A and will be denoted by

(v) $$(\boldsymbol{A})\int_A f\, d\mu.$$

It follows immediately from the above mentioned definition and from the consideration of part I that the integral (v) so defined in a σ-complete Boolean algebra possesses all the properties of the integral in an abstract space. All theorems on the integral proved in Chapter I of Saks [1] are true also in the case of the integral in a σ-complete Boolean algebra. We must only modify accordingly several definitions from the theory of the integral in an abstract space[15].

In particular, if a homomorphism f possesses a definite integral $(\boldsymbol{A},\mu)$ over $A\epsilon\boldsymbol{A}$, then

(vi) $$(\boldsymbol{A})\int_A f\, d\mu=(\boldsymbol{A})\int_A \overset{\circ}{f}\, d\mu-(\boldsymbol{A})\int_A \underset{\circ}{f}\, d\mu.$$

There exists a non-decreasing[16] sequence $\{f_n\}$ of simple non-negative homomorphisms, such that $\overset{\circ}{f}=\lim f_n$. If $R_n=(r_1,r_2,\ldots,r_{m_n})$ is a set of finite real numbers, such that $f_n(R_n)=|\boldsymbol{A}|$, we have

$$(\boldsymbol{A})\int_A f_n\, d\mu=\sum_{i=1}^{m_n}\mu(A\cdot f(r_i))\cdot r_i.$$

Hence $(\boldsymbol{A})\int_A f_n d\mu$ does not depend on the choice of the iso-

[14]) The terminology and notation from the theory of the integral in an abstract space are in this paper the same as in Saks [7].

[15]) For instance, instead of additive functions of a set we must consider additive functions of an element of $\boldsymbol{A}$, instead of measurable functions we must consider always homomorphisms of $\boldsymbol{B}$ in $\boldsymbol{A}$, etc.

[16]) A sequence $\{f_n\}$ is non-decreasing if $f_{n+1}-f_n$ is non-negative $(n=1,2,\ldots)$.

morphic σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$. By Lebesgue's theorem on the integration of monotone sequences[17]), we find that the integral

$$(\boldsymbol{A})\int_A \overset{\circ}{f}\, d\mu=\lim_n (\boldsymbol{A})\int_A f_n\, d\mu$$

does not depend on $\boldsymbol{X}/\boldsymbol{I}$. Analogously we infer that $(\boldsymbol{A})\int_A \underset{\circ}{f}\, d\mu$, and consequently, by (vi), also the integral (v), do not depend on the isomorphic algebra $\boldsymbol{X}/\boldsymbol{I}$.

We must still prove that in the case where the Boolean algebra under consideration is a σ-field $\boldsymbol{X}$ of subsets of an abstract space $\mathscr{X}$ the definition (v) coincides with the usual definition of the integral in the abstract space $\mathscr{X}$. As a σ-quotient algebra which is isomorphic to $\boldsymbol{X}$, we may select the algebra $\boldsymbol{X}/\boldsymbol{0}$ where $\boldsymbol{0}$ denotes the ideal containing one element only: the empty set. Then $[X]=(X)$ and $\bar{\mu}(X)=\mu(X)$ for each $X\epsilon\boldsymbol{X}$. By condition (ii) we obtain that the equation

(vii) $$f(B)=\varphi^{-1}(B) \quad \text{for every } B\epsilon\boldsymbol{B}$$

establishes a one-one correspondence between the class of all homomorphisms f of $\boldsymbol{B}$ in $\boldsymbol{X}$ and the class of all measurable $(\boldsymbol{X})$ functions. By definition (v)

$$(\boldsymbol{X})\int_X \varphi\, d\mu=(\boldsymbol{X})\int_X f\, d\mu,$$

where $f=\varphi^{-1}$. With respect to the correspondence (vii) the definition (v) is in fact a generalization of the definition of the integral in an abstract space.

REFERENCES

[1] G. Birkhoff, *Lattice theory*, American Mathematical Society Colloquium Publications 25 (1940).

[2] A. Bischof, *Beiträge zur Carathéodoryschen Algebraisierung des Integralbegriffs*. Schriften des Mathematischen Instituts der Universität Berlin 5 (1941), Heft 4.

[3] C. Carathéodory, *Entwurf für eine Algebraisierung des Integralbegriffs*. Sitzungsberichte Bayer. Akad. Wiss. 1938, p. 27-38.

[4] L. H. Loomis, *On the representation of σ-complete Boolean algebras*. Bull. Amer. Math. Soc. 53 (1947) pp. 857-760.

[17]) See Saks [7], p. 28.

[5] J. M. H. Olmsted, *Lebesgue theory on a Boolean algebra*, Transactions of the American Mathematical Society 51 (1942), p. 164-193.

[6] J. Ridder, *Zur Mass- und Integrationtheorie in Strukturen*, Indigationes Mathematicae 8 (1946), p. 64-81.

[7] S. Saks, *Theory of the integral*, Monografie Matematyczne, Warszawa—Lwów 1937.

[8] R. Sikorski, *On the representation of Boolean algebras as fields of sets*, Fundamenta Mathematicae 35 (1948), p. 247-258.

[9] R. Sikorski, *On the inducing of homomorphisms by mappings*, Fundamenta Mathematicae 36 (1949), p. 7-22.

ON AN UNSOLVED PROBLEM FROM THE THEORY OF BOOLEAN ALGEBRAS

BY

R. SIKORSKI (WARSAW)

Let $\boldsymbol{X}$ and $\boldsymbol{I}$ be respectively a σ-field[1]) and a σ-ideal of subsets of a set $\mathcal{X}$. It is well known that the quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ is σ-complete. In some cases $\boldsymbol{X}/\boldsymbol{I}$ is further a complete [2]) Boolean algebra. The latter is true, for instance, in two following cases, where:

(a) $\boldsymbol{X}$ is a σ-field on which an enumerably additive finite measure μ is defined, and $\boldsymbol{I}$ is the σ-ideal of all sets of measure zero [3]);

(b) $\boldsymbol{X}$ is the σ-field of all subsets of a topological space[4]) $\mathcal{X}$ which possess the property of Baire [5]) (or: $\boldsymbol{X}$ is the σ-field of all Borel subsets[6]) of $\mathcal{X}$, and $\boldsymbol{I}$ is the σ-ideal of all subsets of the first category in $\mathcal{X}$.

Another kind of σ-field which is often considered beside the fields of measurable or Borel sets is the field $\boldsymbol{S}(\mathcal{X})$ of all subsets of an abstract set $\mathcal{X}$. If $\boldsymbol{I}$ is a *principal* ideal, i. e. if $\boldsymbol{I}$ is formed of all subsets of a set $X \subset \mathcal{X}$, then obviously $\boldsymbol{S}(\mathcal{X})/\boldsymbol{I} = \boldsymbol{S}(\mathcal{X}-X)$ is complete.

---

[1]) Terminology and notation are in this paper the same as in my paper *The integral in a Boolean algebra*, this fascicle, p. 20-21.

[2]) A Boolean algebra $\boldsymbol{A}$ is called *complete*, if for every class $A_0 \subset \boldsymbol{A}$ there exists the least element containing all elements $A \in A_0$.

[3]) More generally: *let $\boldsymbol{I}$ be a σ-ideal of a σ-field $\boldsymbol{X}$; if every class of disjont sets $X \in \boldsymbol{X}-\boldsymbol{I}$ is enumerable, $\boldsymbol{X}/\boldsymbol{I}$ is complete.*

[4]) A space is called *topological*, if it fulfils the four well-known axioms of Kuratowski. See e. g. P. Alexandroff und H. Hopf, *Topologie*, Berlin, 1935, p. 37.

[5]) A subset X of a topological space possesses the *property of Baire* if it can be represented in the form $X = G + P - R$, where G is open, and P and R are of the first category. See C. Kuratowski, *Topologie I*, Monografie Matematyczne, Warszawa-Lwów 1933, p. 49.

[6]) If $\boldsymbol{X}_1$ is the field of all subsets with the property of Baire, $\boldsymbol{X}_2$ is the field of Borel sets, and $\boldsymbol{I}$ is the ideal of sets of the first category, then the algebras $\boldsymbol{X}_1/\boldsymbol{I}$ and $\boldsymbol{X}_2/\boldsymbol{I}$ are isomorphic.