ON THE SYSTEMS OF TOURNAMENTS

BY

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The problem considered in this paper was formulated by H. Steinhaus in 1929 and solved by J. Schreier in 1932 [1]). The solution proposed in the present paper is simpler than that of Schreier.

In each match in a tournament, say a lawn-tennis tournament, two persons play. There are no draws.

We introduce the relation: A is a better player than B. We take this relation as transitive and asymmetric, and as existing between each two players in the tournament. This relation, then, orders the set of the players. We put that the player who wins a match is always a better player than the other. The champion of the tournament will be called the player who is better than all others, the second champion — the player who is better than all others except the champion.

The number of matches necessary for finding the champion or the champion and the second champion depends on the system of tournament and on the results of the matches. Unsatisfactory are called those results which, given a system of tournament, necessitate the highest number of matches for finding the champion or the champion and the second champion. The best system of tournament will be called a system which in the case of unsatisfactory results of the matches makes it possible to establish the champion or the champion and the second champion after a number of matches not higher than the number necessary with any other system.

We propose a tournament system, termed system S, which will be proved to be the best. For this purpose we need the notion of round. A match belongs to the k-th round if one of the players has taken part exactly in $k-1$ matches, and the other in not more than $k-1$ matches.

[1]) J. Schreier, *O systemach eliminacji w turniejach*, Mathesis Polska 7 (1932), p. 154-160.

In order to establish the champion of the tournament by means of the system S, if the number of the players is odd, in each round all players take part who have not lost in one of the preceding rounds, and if the number of the players is even, then in each round all such players take part as in the former case except one. The player who wins the last round is the champion of the tournament for each one of the remaining players has lost one match. In order to establish the second champion an additional number of system S encounters is fought out by those players who have lost in the encounter with the champion.

The player who wins these additional matches is the second champion for each one of the remaining players except the champion has lost a match with a player who is not the champion of the tournament.

Lemma 1. *If n is the number of the players in the tournament then always $n-1$ matches are sufficient to establish the champion of the tournament by the system S.*

Proof. For $n=2$ the Lemma is evidently true. Let k be any natural number >2. We suppose that for each natural i satisfying the inequality

$$2 \leqslant i < k \tag{1}$$

the Lemma is true.

Let s be the number of the matches of the first round of the tournament in which k players take part. Then in the following rounds the number of the players will be $k-s$. It is easily seen that the system of the following matches is the same system S applied to establishing the champion from among $k-s$ players. In virtue of the inductive hypothesis the number of these matches is $k-s-1$. The total number of the matches necessary for establishing the champion from among k players is then

$$s+(k-s-1)=k-1,$$

i. e. the number required by Lemma. This completes the inductive proof.

Lemma 2. *If n is the number of the players in the tournament then $n-1+E\lg_2(n-1)$ matches will be always sufficient to establish the champion and the second champion of the tournament by the system S* [2]).

[2]) where Ex denotes the integral part of x.

Proof. For $n=2$ Lemma is evidently true. Let k be any natural number >2. We suppose that for each natural i satisfying the inequality (1) Lemma is true. Let s be the number of the matches in the first round of the tournament; in the tournament k players take part. In the remaining rounds, therefore, $k-s$ players will take part. Then

$$s=\begin{cases}\dfrac{k}{2} & \text{for } k \text{ being an even number,}\\[2mm] \dfrac{k-1}{2} & \text{for } k \text{ being an odd number.}\end{cases} \tag{2}$$

The following matches differ from the system S applied to establishing the champion and the second champion from among $k-s$ players only by the circumstance that there may be an additional match of the player who lost in his encounter with the champion of the tournament in the first round against one of the players who lost in their encounters with the champion in the following rounds.

In virtue of the inductive hypothesis the number of the matches necessary for establishing the champion and the second champion from among $k-s$ players is

$$k-s-1+E\lg_2(k-s-1).$$

Then the number of matches which is always sufficient for establishing the champion and the secoud champion by the system S is

$$s+(k-s-1+E\lg_2(k-s-1))+1=k-1+E\lg_2(k-1),$$

because by (2)

$$E\lg_2(k-s-1)+1=E\lg_2(k-1).$$

This completes the inductive proof of Lemma.

Lemma 3. If n is the number of the players in the tournament, then $n-1$ is the least number of matches which in case of unsatisfactory results of the matches is sufficient for establishing the champion of the tournament.

Proof. Lemma 3 results from Lemma 1 and from the remark that for the champion of the tournament to be established each player except the champion himself must loose at least one match.

Lemma 4. If n players take part in the tournament then $n-1+E\lg_2(n-1)$ is the least number of matches which in the case of unsatisfactory results of the matches is sufficient for establishing the champion and the second champion of the tournament.

Proof. It will be proved that in some cases (depending upon the results of drawing lots) the number of matches necessary for establishing the champion and the second champion is not smaller than the number mentioned in the Lemma, regardless of the system of tournament.

For $n=2$ Lemma obviously holds.

Let k be any natural number >2. Let us suppose that the lemma holds for every natural i satisfying the inequality (1). Let s be the number of the matches of the first round of the tournament in which k players take part. In the following matches at least $k-s$ players take part. We suppose that each player who has lost in the first round is worse then any player who has not lost in that round. So the following matches played by the players of one of these groups against the players of the other will not permit of any conclusions as to the ordering of the set of the players who have not lost in the first round. The results of the first round do not allow of conclusions as to the ordering of the players of that set. For establishing the champion and the second champion of the tournament all the matches will be necessary which are necessary for establishing the champion and the second champion of those players who have not lost in the first round. According to the inductive hypothesis there are at least

$$k-s-1+E\lg_2(k-s-1)$$

such matches.

Besides, we can assume that the champion of the tournament has played in the first round. Then, for establishing the second champion a match is necessary fought out by the player who has lost his encounter with the champion in the first round against a player who is not the champion of the tournament.

Thus the number of matches necessary for establishing the champion and the second champion of the tournament if the results of the encounters are unsatisfactory is not smaller than

$$s+[k-s-1+E\lg_2(k-s-1)]+1;$$

and by the inequality $s \leqslant k-s$, it is also not smaller than $k-1+E\lg_2(k-1)$. Hence and from Lemma 2 follows Lemma 4.

From the lemmas proved above follows

Theorem. The best system of tournament the purpose of which is to establish the champion and the second champion is the system S.

The number of matches sufficient for establishing the champion of the tournament by this system is $n-1$; the number sufficient for establishing the champion and the second champion is $n-1+E\lg_2(n-1)$.

ARITHMETICS OF NATURAL NUMBERS AS PART OF THE BI-VALUED PROPOSITIONAL CALCULUS

BY

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This paper[1]) contains an outline of a method of constructing — within the bi-valued propositional calculus — of an arithmetic of natural numbers, elementary and extremely narrow, but sufficient for practical purposes. Utilization of such a logical construction to the theory of relay and electronic digital machines will be discussed elsewhere.

I wish to express my thanks to J. Egerváry, Budapest, for his valuable suggestions which helped me to formulate the definition scheme of addition of natural numbers, as well as to A. Mostowski, Warsaw, for certain suggestions of general character.

1. *Falsum, verum,* negation, implication, alternation, conjunction, as well as existential and general quantifiers, are symbolised, respectively, as follows:

$$0,\ 1,\ ',\ \rightarrow,\ \dot{+},\ .,\ \Sigma,\ \Pi\ ^{2)}.$$

The letters

$$p, p_{\mathrm{I}}, p_{\mathrm{II}}, p_{\mathrm{III}}, \ldots, q, q_{\mathrm{I}}, q_{\mathrm{II}}, q_{\mathrm{III}}, \ldots,$$

$$r, r_{\mathrm{I}}, r_{\mathrm{II}}, r_{\mathrm{III}}, \ldots, s, s_{\mathrm{I}}, s_{\mathrm{II}}, s_{\mathrm{III}}, \ldots, t, t_{\mathrm{I}}, t_{\mathrm{II}}, t_{\mathrm{III}}, \ldots,$$

are propositional variables.

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[1]) Partly identical with the author's paper, *Les tautologies arithmétiques du calcul propositionnel et les circuits éléctriques*, read at the First Congress of Hungarian Mathematicians (Budapest 1950).

[2]) The method of defining Σ, Π in the language of the propositional calculus was formulated in the author's paper *Functors of the Propositional Calculus*, read at the Sixth Congress of Polish Mathematicians, Warsaw, September 20-23, 1948, see *VI Zjazd Matematyków Polskich*, Dodatek do Rocznika Polskiego Towarzystwa Matematycznego 22 (1950), p. 78-80.