AN ALGEBRAIC PROOF OF COMPLETENESS FOR THE TWO-VALUED PROPOSITIONAL CALCULUS

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The chief result of the propositional calculus is the theorem of his completeness, which says that *every formula satisfying the 0-1-truth-table of Schröder derives from accepted axioms.*

By a *truth-table* we mean a quadruple $\boldsymbol{M}=\langle A,B,\rightarrow,{}^{-}\rangle$, where A and B are sets, $B\subset A$, and $\rightarrow$, ${}^{-}$, are functions of two and of one variable defined on A with values in A. The set B is called *set of designated values*, the functions $\rightarrow$ and ${}^{-}$ are interpretations for implication and negation, respectively. The 0-1-truth-table is defined in the section 2.

Many proofs of the theorem mentioned above are known; the first of them is due to Łukasiewicz [1]), the simplest ones are that of Hilbert [2]) and that recently published by Henkin [3]).

In this paper we give a proof of this theorem which differs from the others by its algebraical method. We use the known construction of Lindenbaum [4]), which gives for each system of propositional calculus a truth-table, whose elements are formulae, and which is adequate to this system, and then we prove that if a formula a does not fulfil this truth-table, then a homomorphism h of this truth-table on the 0-1-truth-table exists, so that $h(a)=0$. The existence of this homomorphism enables us to conclude that a does not fulfil the 0-1-truth-table, which completes the proof.

[1]) J. Łukasiewicz, *Elementy logiki matematycznej*, Warszawa 1929.

[2]) D. Hilbert und W. Ackermann, *Grundzüge der theoretischen Logik*, Berlin 1928, p. 3-31.

[3]) L. Henkin, *Fragments of the propositional calculus*, Journal of Symbolic Logic 14 (1949), p. 42-48.

[4]) J. C. C. McKinsey, *A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology*, Journal of Symbolic Logic 6 (1941), p. 117-134, especially p. 122.

The truth-table $\boldsymbol{M}=\langle A,B,\rightarrow,{}^{-}\rangle$ is normal, if a, $a\rightarrow b\,\epsilon\,B$, implies $b\,\epsilon\,B$, and is called *truth-table of two-valued propositional calculus* if it is adequate for the system of this calculus (i.e. $E(\boldsymbol{M})=E(\boldsymbol{B})$, *vide* 2). It is known that every normal truth-table of the two-valued propositional calculus may be considered as a Boolean algebra, and therefore the 0-1-truth-table is the algebra of subsets of a one-point set. Hence it is clear that this proof is closely connected with the theorem of M. H. Stone [5]):

For every element a of a Boolean algebra $\boldsymbol{A}$ (with exception of the greatest element in $\boldsymbol{A}$), there exists a homomorphism h of $\boldsymbol{A}$ into the one-point set algebra, such that $h(a)=0$ (0 — the empty set).

1. System $\boldsymbol{L}$. We denote by S the set of all well formed formulae which consists of the signs: $\rightarrow$ (implication), ${}^{-}$ (negation), $p,q,r,s,\ldots$ (variables), and parentheses. By $L\subset S$ we denote the system of those formulae which follow (using the rules of substitution and derivation) from the five axioms:

$$(A_1)\qquad [p\rightarrow(q\rightarrow r)]\rightarrow[(p\rightarrow q)\rightarrow(p\rightarrow r)],$$

$$(A_2)\qquad p\rightarrow(q\rightarrow p),$$

$$(A_3)\qquad p\rightarrow p,$$

$$(A_4)\qquad (\bar{p}\rightarrow p)\rightarrow p,$$

$$(A_5)\qquad p\rightarrow(\bar{p}\rightarrow q).$$

2. Truth-tables $\boldsymbol{B}$ and $\boldsymbol{L}$. The truth-table

$$\boldsymbol{B}=\langle(0,1),(1),\Rightarrow,{}^{=}\rangle$$

in which the functions $\Rightarrow$ and ${}^{=}$ are defined by the equations

$$(2.1)\qquad (0\Rightarrow 0)=(0\Rightarrow 1)=(1\Rightarrow 1)=(\overline{\overline{0}}=1),$$

$$(2.2)\qquad (1\Rightarrow 0)=(\overline{\overline{1}}=0),$$

is called *0-1-truth-table of Schröder*. By

$$\boldsymbol{L}=\langle S,L,\rightarrow,{}^{-}\rangle$$

we denote the adequate Lindenbaum's truth-table for the system L, and thus

(2.3) *the value of the binary function $\rightarrow$ for two formulae $a,\beta\,\epsilon\,S$, is the formula $a\rightarrow\beta$ (implication with the antecedent a and the consequent β),*

[5]) M. H. Stone, *The theory of representations for Boolean algebras*, Transactions of the American Mathematical Society 40 (1936), p. 78.

(2.4) *the value of the function* $\bar{\ }$ *for the formula* α *is the formula* $\bar{\alpha}$ *(negation of* α*).*

The signs $\to$ and $\bar{\ }$ are thus used in both meanings: as primitive signs in the formulae and as signs of functions in the truth-table **L**, but this does not cause any misunderstanding. Should we, to avoid this equivocality, denote the functions of **L** by $\dashrightarrow$ and $\ddot{\ }$, we could write down their definitions:

(2.3*) $$\alpha \dashrightarrow \beta = \alpha \to \beta,$$

(2.4*) $$\ddot{\alpha} = \bar{\alpha}.$$

If **M** is a truth-table, then we denote by $E(\boldsymbol{M})$ the set of all those formulae which fulfil **M**. From the theorem of Lindenbaum it follows that **L** is adequate for the system L, which means

(2.5) $$L = E(\boldsymbol{L}).$$

If h maps S into $(1,0)$, we say that h is a homomorphism of **L** into **B**, if for $\beta, \gamma \in S$

(2.6) $$h(\beta) \Rightarrow h(\gamma) = h(\alpha \to \beta),$$

(2.7) $$\overline{\overline{h(\beta)}} = h(\bar{\beta}).$$

It is easy to see that

(2.8) if h is a homomorphism of **L** into **B**, $\alpha \in S$, and $h(\alpha)=0$, then α non-$\in E(\boldsymbol{B})$.

3. Theorem of completeness. This theorem can be expressed in the terminology of the sections 1 and 2 as follows:

(3.1) $$L = E(\boldsymbol{B}),$$

or equivalently, in view of (2.5),

(3.2) $$E(\boldsymbol{L}) = E(\boldsymbol{B}).$$

It is obvious that

(3.3) $$L \subset E(\boldsymbol{B});$$

therefore in order to prove (3.1) it is sufficient to show that

(3.4) $$E(\boldsymbol{B}) \subset L.$$

4. Lemma. *If* $\alpha \in S - L$, *there exists a set* $I \subset S$ *such that*

(4.1) $\beta \in I$ *and* $\beta \to \gamma \in I$ *imply* $\gamma \in I$,

(4.2) $$L \subset I,$$

(4.3) $$\alpha \text{ non-}\in I,$$

(4.4) *for every* $\beta \in S$, *either* $\beta \in I$ *or* $\bar{\beta} \in I$.

Proof. If $A \subset S$ and $\beta \in S$, we denote by $A(\beta)$ the set of all $x \in S$ such that $\beta \to x \in A$.

Let α non-$\in L$, and let $\alpha_1, \alpha_2, \ldots$ be the sequence of all elements of S. We shall define by induction an increasing sequence of sets as follows:

(4.5) $$A_0 = L(\alpha),$$

(4.6) $$A_{n+1} = \begin{cases} A_n & \text{if } \alpha_{n+1} \in A_n, \\ A_n(\alpha_{n+1}) & \text{if } \alpha_{n+1} \text{ non-}\in A_n, \end{cases}$$

and we set

(4.7) $$I = \sum_{i=1}^{\infty} A_i.$$

Every set A_i has the properties (4.1), (4.2) (for $I = A_i$), because L has these properties (for $I = L$) and if A has these properties and α_i non-$\in A$, then for $A(\alpha_i)$ the property (4.1) results from (A_1), (4.2) from (A_2). Moreover from (A_2) it follows that $A_n \subset A_{n+1}$ and this enables us to conclude that $I = \sum_{n=1}^{\infty} A_n$ has both these first properties. From (A_3) we have $\bar{\alpha} \in A_0 = L(\alpha)$; therefore $\bar{\alpha} \in A_n$. Suppose that for some n, $\alpha \in A_{n+1} = A_n(\alpha_{n+1})$; then (A_5) implies $A_{n+1} = S$ and by definition $\bar{\alpha}_{n+1} \to \alpha_{n+1} \in A_n$; therefore in view of (A_4) we have $\alpha_{n+1} \in A_n$, which contradicts (4.6); this proves that α non-$\in A_n$ for any n and (4.3) holds for I.

Finally, if $\alpha_{n+1} \in A_n$, then $\alpha_{n+1} \in I$, if α_{n+1} non-$\in A_n$, then by (4.6) and (A_3), $\bar{\alpha}_n \in A_n \subset I$. This proves (4.4) for I, q. e. d.

5. Proof of (3.4). Let α non-$\in L = E(\boldsymbol{L})$; then by lemma there exists a set I with properties (4.1)-(4.4). We have:

(5.1) If $\beta \in S$, $\gamma \in I$, then $\beta \to \gamma \in I$;

this follows from (4.1), (4.2) and A_2.

(5.2) If $\beta \in S - I$, $\gamma \in S$, then $\beta \to \gamma \in I$;

this follows from (4.1), (4.2), (4.4) and (A_5).

(5.3) If $\beta \in I$, $\gamma \in S - I$, then $\beta \to \gamma \in S - I$;

this follows from (4.1).

(5.4) If $\beta \in I$, then $\bar{\beta} \in S - I$; if $\beta \in S - I$, then $\bar{\beta} \in I$;

this follows from (4.6).

These four properties show that the mapping

(5.5) $$h(x)=\begin{cases}0 & \text{if } x\epsilon S-I,\\ 1 & \text{if } x\epsilon I\end{cases}$$

is a homomorphism of $\boldsymbol{L}$ into $\boldsymbol{B}$, and in view of (4.3),

(5.6) $$h(\alpha)=0.$$

From (5.5), (5.6) and (2.8), we have α non-$\epsilon E(\boldsymbol{B})$, q. e. d.

6. Remarks. It is easy to see that this proof can be carried out without the use of the construction of Lindenbaum and the truth-table $\boldsymbol{L}$. We have introduced these notions rather in order to emphasize the algebraic method of this proof.

The axioms (A_1)-(A_5) are not independent, but only (A_3) is dependent from the others. Therefore the axioms (A_1), (A_2), (A_4), and (A_5), form an independent and complete set of postulates for the two-valued propositional calculus.

SUR LA CONVERGENCE STATISTIQUE *

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Soit $\{k_n\}$ une suite croissante de nombres naturels. Désignons par i_n le nombre des termes $k_j \leqslant n$. Si la limite $\lim i_n/n$ existe nous l'appellons *fréquence* de $\{k_n\}$ dans la suite de tous les nombres naturels (ou fréquence de $\{k_n\}$, tout court). $W(x)$ étant une fonction propositionnelle et $\{a_n\}$ étant une suite, nous désignerons par $\underset{n}{\mathrm{fr}}\,[W(a_n)]$ la fréquence de la suite des nombres n pour lesquels $W(a_n)$ est vérifiée.

Définition 1. Une suite $\{a_n\}$ de nombres réels est dite *mesurable* si $\underset{n}{\mathrm{fr}}\,[a_n<a]$ existe pour tout a sauf pour les valeurs exceptionnelles qui constituent un ensemble au plus dénombrable.

Définition 2. Nous disons que la suite $\{a_n\}$ *converge statistiquement vers* a si elle est mesurable et si, pour tout $\varepsilon>0$, on a $\underset{n}{\mathrm{fr}}\,[|a_n-a|\geqslant\varepsilon]=0$. Nous écrivons alors $\lim\operatorname{stat} a_n=a$.

Evidemment

(i) la condition $\lim\operatorname{stat} a_n=a$ équivaut à ce qu'il existe une suite $\varepsilon_j\to 0$, $\varepsilon_j>0$ telle que $\underset{n}{\mathrm{fr}}\,[|a_n-a|>\varepsilon_j]=0$ pour tout j,

(ii) les théorèmes élémentaires sur la somme, la différence, le produit et le quotient de deux suites convergentes sont aussi valables pour les suites statistiquement convergentes.

Nous aurons encore besoin de la proposition suivante qu'on prouve sans difficulté:

(iii) pour une suite bornée $\{a_n\}$ de nombres non négatifs, la condition $\lim\operatorname{stat} a_n=0$ équivaut à $\lim \frac{1}{n}\sum_{i=1}^{n} a_i=0$.

* La démonstration primitive du théorème de la page 242 présentée le 18 février 1949 par H. Steinhaus à la Section de Wrocław de la Société Polonaise de Mathématique (cf. ce volume, p. 73) est remplacée ici par une version simplifiée et basée sur les idées de A. Zygmund et les miennes.