D'après (8.3), Ψ équivaut à

(8.5) $$[\prod_{x,y}(x=y)]+Q_1\underset{x_1}{}\dots\underset{x_n}{Q_n}\Psi$$

et, d'après (8.4), nous pouvons remplacer dans (8.5) chaque inégalité par une fonction propositionnelle positive, et nous obtenons enfin une proposition positive.

TRAVAUX CITÉS

[1] G. Birkhoff, *Lattice Theory*, New York 1948.

[2] — *On the structure of abstract algebras*, Proceedings of the Cambridge Philosophical Society 31 (1935), p. 433-454.

[3] — and S. MacLane, *A survey of modern Algebra*, New York 1946.

[4] N. Bourbaki, *Eléments de mathématique, Algèbre*, chapitre I, Paris 1942.

[5] L. Byrne, *Two brief formulations of Boolean Algebra*, Bulletin of the American Mathematical Society 52 (1946), p. 269-272.

[6] A. Horn, *On sentences which are true of direct union of algebras*, Journal of Symbolic Logic 16 (1951), p. 14-21.

[7] E. V. Huntington, *Sets of independent postulates for the Algebra of Logic*, Transactions of the American Mathematical Society 5 (1904), p. 288-309.

[8] W. Sierpiński, *Algèbre des Ensembles*, Warszawa-Wrocław 1951.

Institut Mathématique de l'Université de Wrocław

REMARKS ON BOOLEAN ALGEBRAS

BY

M. KATĚTOV (PRAGUE)

The present note* contains an example of a Boolean algebra without proper automorphisms[1]) and a sufficient and necessary condition[2]) for a Boolean algebra to be a Hausdorff space in its interval topology[3]). The example mentioned above will be derived from the theory of the Čech compactification. If

1° S is a *completely regular space* (i.e. a Hausdorff space such that, for any closed set $M\subset S$ and any $x\in S-M$, there exists a continuous real-valued function f in S such that $f(x)=1$, $f(z)=0$ (whenever $z\in M$)), R is compact (=bicompact), $R\supset S$, $R=\overline{S}$,

2° for any bounded continuous function f in S there exists a continuous function F in R which coincides with f in S,

then R is called the *Čech compactification*[4]) of S and is denoted by βS.

Lemma 1. *If P is completely regular, βP denotes its Čech compactification, $x\in\beta P-P$, and there exist open sets $G_n\subset\beta P$ such that $x\in G_n$, $P\prod_{n=1}^{\infty}G_n=0$, then there exists no sequence of different points $x_n\in\beta P$ converging to x.*

Proof. Suppose, on the contrary, that $x_n\in\beta P$, $x_n\to x$, $x\in\beta P-P$, $x_m\neq x_n\neq x$ whenever $m\neq n$. Put

$$P_1=P+\sum_{n=1}^{\infty}(x_n)+(x).$$

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[1]) Cf. Birkhoff [1], p. 162, Problem 74. (Numbers in brackets refer to the list at the end of the paper).

[2]) Cf. Birkhoff [1], p. 62, Problem 23.

[3]) Birkhoff [1], p. 60.

[4]) Cf. Čech [2].

Obviously, there exist open (in P_1) sets H_n such that

$$\prod_{n=1}^{\infty} H_n = (x).$$

Then there exist continuous functions h_n $(n=1,2,\ldots)$ in P_1 such that $0 \leqslant h_n(z) \leqslant 2^{-n}$ for any $z \in P_1$, $h_n(x)=0$, $h_n(z)=2^{-n}$ if $z \in P_1 - H_n$.

Put $h=\sum_{n=1}^{\infty} h_n$. Then h is a function continuous in P_1, $0 \leqslant h(z) \leqslant 1$ for any $z \in P_1$, $h(x)=0$, $h(z) \neq 0$ if $z \neq x$. Consequently, $h(x_n) \to 0$, $h(x_n) \neq 0$. Obviously, there exist subsequences $\{x_{k_n}\}$, $\{x_{l_n}\}$, such that we never have $h(x_{k_m}) = h(x_{l_n})$. Then there exists a bounded function φ continuous in $(0,1]$ such that

$$\varphi(h(x_{k_m}))=0, \qquad \varphi(h(x_{l_n}))=1 \qquad (m,n=1,2,\ldots).$$

Put, for $z \in P$, $f(z)=\varphi(h(z))$, and let F be the continuous extension of f over βP. It is easy to see that $F(x_{k_m})=0$, $F(x_{l_n})=1$; since $x_{k_m} \to x$, $x_{l_n} \to x$ we have $F(x_{k_m}) \to F(x)$, $F(x_{l_n}) \to F(x)$, and therefore $F(x)=0$, $F(x)=1$. This proves the lemma.

Lemma 2. *There exists a denumerable normal space P such that*

(i) *there exists no homeomorphism of P onto itself except the identity mapping;*

(ii) *for any $x \in P$, there exists a sequence of points $x_n \in P$, $x_n \neq x$, which converges to x.*

Proof. Let M denote the space of rational numbers. Consider the space $T=\beta M \times \beta M$ and put $S=M \times M$. For any $x \in T$, let $\Phi(x)$ denote the set of all points $y=\varphi(x)$, where φ is a continuous mapping of $S+(x)$ into T such that $\varphi(S)=S$.

Since the set of all mappings of S into S has power $\leqslant \mathfrak{c}$, and $\bar{S}=T$, it is clear that every $\Phi(x)$ has power $\leqslant \mathfrak{c}$ ($\mathfrak{c}$ denotes the power of the continuum).

Let $\{G_n\}$ denote a countable open base of M. It is well-known[5]) that every $\bar{G}_n$ (closure in βM) has power $2^{\mathfrak{c}}$. Therefore it is easy to prove by induction that there exist points $\xi_n \in \beta M - M$, $\eta_n \in \beta M - M$ $(n=1,2,\ldots)$ such that

[5]) Cf. e. g. Pospišil [4].

(a) $\xi_n \in \bar{G}_n$, $\eta_n \in \bar{G}_n$;

(b) if ξ, ξ', η, η', are points from M, then, for $m,n=1,2,\ldots$, $m<n$, (ξ_n,η') non $\in \Phi(\xi_m,\eta)$, (ξ_n,η') non $\in \Phi(\xi,\eta_m)$, (ξ',η_n) non $\in \Phi(\xi,\eta_m)$, (ξ',η_n) non $\in \Phi(\xi_m,\eta)$, and, moreover, (ξ,η_n) non $\in \Phi(\xi_n,\eta)$, for any $n=1,2,\ldots$

Let A_n $(n=1,2,\ldots)$ denote the set of all (ξ_n,η), $\eta \in M$, and let B_n $(n=1,2,\ldots)$ denote the set of all (ξ,η_n), $\xi \in M$. Then by (b) we have, for any $n=1,2,\ldots$,

$$(*) \qquad \begin{aligned} (A_n+B_n)\cdot\Phi(x)&=0, \quad \textit{whenever} \quad x \in A_m+B_m,\ m<n; \\ B_n\Phi(x)&=0, \quad \textit{whenever} \quad x \in A_n. \end{aligned}$$

Put $Q=\sum_{n=1}^{\infty} A_n + \sum_{n=1}^{\infty} B_n$, $P=S+Q$. It is well-known (and follows e. g. from Lemma 1) that no $\xi \in \beta M - M$ satisfies in βM the first denumerability axiom[6]). It is well-known, too, that if X is a regular space, $x \in Y \subset X$, $\bar{Y}=X$, then x satisfies the first denumerability axiom in Y if and only if it satisfies this axiom in X. (The first half ("if") of this assertion being trivial, let x satisfy the axiom in Y; let U_n be open in Y, $x \in U_n$, and suppose that every open (in Y) set V such that $x \in V$ contains some U_n. Put $H_n = X - \overline{X - \bar{U}_n}$; every H_n is open in X and contains x. Let G be open in X, $x \in G$. Since X is regular, there exists an open set $G_1 \subset X$ such that $x \in G_1 \subset \bar{G}_1 \subset G$. Choose U_n such that $U_n \subset G_1 Y$. Then $x \in H_n \subset \bar{U}_n \subset \subset \overline{G_1 Y} = \bar{G}_1 \subset G$. This proves the assertion.)

Therefore, S being dense in P, it is easy to see that

$(\ast\ast)$ *point $x \in P$ satisfies the first denumerability axiom if and only if $x \in S$.*

Now let f be a homeomorphism of P onto P. Then, by $(\ast\ast)$, $f(S)=S$ and therefore $f(x) \in \Phi(x)$, $x \in \Phi(f(x))$, for any $x \in P$. This implies, by $(*)$, that $f(A_n)=A_n$, $f(B_n)=B_n$, $n=1,2,\ldots$ Suppose that for some $z \in S$, $f(z) \neq z$. Then we have, if $z=(\xi,\eta)$, $f(\xi,\eta)=(\xi',\eta')$, $\eta \neq \eta'$ or $\xi \neq \xi'$.

Suppose e. g. that $\eta \neq \eta'$. Obviously there exists a neighbourhood U of η' (in βM) such that η' non $\in \bar{U}$. Let B denote the sum

[6]) A point x of a Hausdorff space R is said to *satisfy the first denumerability axiom* if there exist open sets U_n $(n=1,2,\ldots)$ containing x and such that, for any open V containing x we have, for some n, $U_n \subset V$.

of all B_n such that $\eta \in \bar{U}$. We have $(\xi, \eta) \in \bar{B}$, for the set of all η_n is dense in βM and therefore every neighbourhood of (ξ, η) contains points (ξ, η_n), $\eta_n \in \bar{U}$. Since f is continuous, we have $f(\xi, \eta) \in \overline{f(B)}$; hence $(\xi', \eta') \in \bar{B}$, for $f(B) = B$. This is a contradiction, for η' non $\in \bar{U}$. Therefore, $f(z) = z$ for any $z \in S$ and hence for any $z \in P$, as S is dense in P. Thus P has property (i). As for property (ii), it is evident, for if e. g. $x = (\xi_n, \eta) \in Q$, then there exist $\eta_k \in M$, $\eta_k \to \eta$, and we have $(\xi, \eta_k) \in P$, $\lim_{k \to \infty} (\xi_n, \eta_k) = x$.

Theorem 1. There exists a Boolean space (i. e. a 0-dimensional compact space) *which admits of no homeomorphism onto itself except identity.*

Proof. Let P have properties given in Lemma 2. Put[7] $R = \beta P$. It is easy to see that R is 0-dimensional. Let f be a homeomorphism of R onto R. Lemma 1 and property (ii) of P imply that $f(S) = S$. Property (i) of P implies that f is the identity mapping.

Theorem 2. There exists an (infinite) Boolean algebra admitting of no proper (i. e. non-identical) *automorphism.*

Proof. Let $\mathfrak{A}$ denote the Boolean algebra of open-and-closed subsets of the space R given above. If φ is an automorphism of $\mathfrak{A}$, then, for any $x \in R$, the intersection of all $A \in \mathfrak{A}$ such that $x \in \varphi(A)$ contains exactly one point which will be denoted by $f(x)$. It is easy to see that f is a homeomorphism of R onto R. Therefore f, and hence φ, is the identical mapping.

Remarks. 1. I do not know whether there exists an (infinite) Boolean algebra $\mathfrak{A}$ having no proper homeomorphism onto itself.

2. The algebra $\mathfrak{A}$ defined above is not σ-complete. I do not know whether there exists a complete (or σ-complete at least) Boolean algebra without (proper) automorphisms.

We shall now consider the interval topology of Boolean algebras[8]). The interval topology of a partially ordered set S is defined by taking sets of the form "all $x \in S$ contained in a (i. e. such that $x \leqslant a$)", "all $x \in S$ containing a (i. e. such that $x \geqslant a$)", as well as the whole set S of course, as a subbase of closed sets (i. e. taking finite intersections of their complements as an open base).

[7]) This is possible, for βP exists for any completely regular P; see Čech [2].

[8]) See Birkhoff [1], p. 60.

Lemma 3. Let L be a distributive lattice with 0, *and let $A \subset L$, $B \subset L$, be finite sets of elements $\neq 0$. Then either there exists $x \in L$ meeting all $a \in A$* (i. e. $x \cap a \neq 0$ if $a \in A$) *and containing no $b \in B$* (i. e. $x \geqslant b$ for no $b \in B$) *or some $a \cap b$, $a \in A$, $b \in B$, contains an atom*[9]).

Proof. Suppose that no such x exists. Let $M \subset L$ consist of all meets, different from 0, of some elements $z \in A + B$ and let N be the set of all minimal $u \in M$ (i. e. of $u \in M$, such that $u > v$ for no $v \in M$). Obviously, any two different elements u_1, u_2, from N are disjoint (i. e. $u_1 \cap u_2 = 0$). Let A^* (or B^* respectively) denote the set of all $u \in N$ which are contained in some $a \in A$ (or in some $b \in B$ respectively). If there existed, for each c belonging both to A^* and B^*, an element $d < c$, $d \neq 0$, then denoting by s the join of all such elements d (one for each c) and of all $v \in A^*$ not belonging to B^* we should have

(i) $s \cap w \neq 0$ if $w \in A^*$,

(ii) $s \geqslant w$ for no $w \in B^*$.

This would imply that s meets each $a \in A$ and contains no $b \in B$, which contradicts the assumption. Therefore there exists an atom c belonging to $A^* B^*$, and hence contained in some $a \cap b$, $a \in A$, $b \in B$.

Theorem 3. If A is a Boolean algebra, $x \in A$, $y \in A$, then x and y have disjoint neighbourhood in the interval topology of A if and only if $(x \cap y') \cup (x' \cap y)$ contains an atom.

Proof. I. If $c \leqslant (x \cap y') \cup (x' \cap y)$ is an atom, suppose that e. g. $c \leqslant x \cap y'$; let G denote the set of $z \in A$ not containing c, and let H denote the set of $z \in A$ not contained in c'. Then GH is empty and it is easy to see that G is a neighbourhood of y, H is a neighbourhood of x.

II. Let x, y, have disjoint neighbourhoods. Then there exist finite subsets (of A) U_x, V_x, U_y, V_y, such that, denoting by G_t ($t = x, y$) the set of $z \in A$ containing no $u \in U_t$ and contained in no $v \in V_t$, we have $x \in G_x$, $y \in G_y$, $G_x G_y = 0$. Let M denote the set consisting of all meeting points $u \cap x'$, $u \in U_x$, and $u \cap y'$, $u \in U_y$; let N denote the set of all meeting points $v' \cap x$, $v \in V_x$, and $v' \cap y$,

[9]) An element c of a lattice L with 0 is called an *atom* if $c \neq 0$ and there exists no $d < c$, $d \neq 0$.

$v \in V_y$. It is easy to see that, since $G_x G_y = 0$, there exists no $z \in A$ meeting all $n \in N$ and containing no $m \in M$. Hence, by Lemma 3, some $m \cap n$, $m \in M$, $n \in N$, contains an atom c. Obviously, we have either $c \leqslant x' \cap y'$ or $c \leqslant x \cap y'$.

Corollary 1. A Boolean algebra is a Hausdorff space in its interval topology if and only if it is atomic.

Corollary 2. A Boolean algebra is a compact Hausdorff space in its interval topology if and only if it is isomorphic with the algebra $2^{\mathfrak{m}}$ of all subsets of some aggregate.

Proof. Sufficiency: $2^{\mathfrak{m}}$ is compact by Frink's [3] theorem, and a Hausdorff space by the above corollary.

Necessity: a lattice compact in its interval topology being complete[10]), we apply Corollary 1 observing that a complete atomic Boolean algebra is isomorphic with $2^{\mathfrak{m}}$.

Remarks. 1. Birkhoff's book contains the following proposition, stated without proof: *any partly ordered set is a Hausdorff space in its order topology*[11]). By Corollary 2, this assertions implies (since a complete lattice is compact in its interval topology) the following proposition: *if the order topology and the interval topology of a complete Boolean algebra coincide, then it is isomorphic with* $2^{\mathfrak{m}}$[12]).

2. It is easy to show that the interval and the order topology of $2^{\mathfrak{m}}$ coincide. For, let $\mathfrak{A}$ be the Boolean algebra of subsets of a given set S. Since any set closed in the interval topology, is closed in the order topology too[11]), we have only to prove: if

$$A \in \mathfrak{A}, \quad \mathfrak{M} \subset \mathfrak{A},$$

and

(*) every neighbourhood of A in the interval topology of $\mathfrak{A}$ intersects $\mathfrak{M}$,

then there exists a directed set $\{X_\alpha\}$, $X_\alpha \in \mathfrak{M}$, which converges to A[13]).

Assumption (*) implies: if $M \subset A$, $N \subset S - A$ are finite sets, then there exists $X = X(M,N) \in \mathfrak{M}$, $M \subset X$, $N \subset S - X$. Let the set of all pairs (M,N), where $M \subset A$, $N \subset S - A$, are finite, be (partially) ordered as follows: (M_1, N_1) precedes (M_2, N_2) if and only if $M_1 \subset M_2$, $N_1 \subset N_2$. It it easy to see that the directed (in the above sense) set $\{X(M,N)\}$ converges to A.

3. There exist complete Boolean algebras which are not compact in the order topology.

Example: let $\mathfrak{A}$ be the Boolean algebra of all measurable subsets of the interval $I = (0,1)$ modulo sets of measure zero. It is well-known[14]) that $\mathfrak{A}$ is complete. It is easy to see that there exist sets $A_{n,i} \subset I$ ($n = 1, 2, \ldots$; $i = 0, 1$) such that $A_{n,1} = I - A_{n,0}$ and

($\substack{*\\*}$) $\prod_{k \in K} A_{k,i(k)}$ has measure 2^{-p}, where $i(k) = 0$ or 1, K is a set of p different natural numbers (we may put $A_{n,0}$ equal to the sum of intervals $(k \cdot 2^{-n}, (k+1) \cdot 2^{-n})$, k pair, $0 \leqslant k < 2^n$).

Let $a_{n,i}$ be the element of $\mathfrak{A}$ corresponding to the set $A_{n,i}$. Then the assertion ($\substack{*\\*}$) implies at once that $\bigcap_{k \in N} a_{k,0} = 0$, $\bigcup_{k \in N} a_{k,0} = 1$, N being an arbitrary infinite set of natural numbers. Therefore no directed set of elements $a_{k,0}$ can converge (except in trivial cases). Hence $\mathfrak{A}$ is not compact.

4. I do not know whether there exists a complete non-atomic Boolean algebra which is compact in its order topology.

[10]) Birkhoff [1], p. 61, Exercise 4b.
[11]) Birkhoff [1], p. 60.
[12]) Birkhoff [1], p. 166, Problem 76.
[13]) In the sense of Birkhoff [1], p. 59-60.

[14]) See e. g. Wecken [5].

REFERENCES

[1] G. Birkhoff, *Lattice Theory*, revised edition, 1948.

[2] E. Čech, *On Bicompact Spaces*, Annals of Mathematics 38 (1937), p. 823-844.

[3] O. Frink, *Topology in Lattices*, Transactions of the American Mathematical Society 51 (1942), p. 569-582.

[4] B. Pospíšil, *A Remark on Bicompact Spaces*, Annals of Mathematics 38 (1937), p. 845.

[5] F. Wecken, *Abstrakte Integrale und fastperiodische Funktionen*, Mathematische Zeitschrift 45 (1939), p. 377-404.