GROUPS CONNECTED WITH BOOLEAN ALGEBRAS

(PARTIAL SOLUTION OF THE PROBLEM P 92)

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The aim of this note is to solve the problem proposed in the remark 5° of the preceding paper [1]) for a special kind of Boolean rings A, namely for the so called *rings with an ordered basis* [2]).

We use symbols defined in the remark 5° of [1]. Besides this we put

$$(a \leqslant b) \equiv (a \cdot b = a), \qquad (a < b) \equiv ((a \neq b) \text{ and } (a \leqslant b)),$$

for arbitrary $a, b \in A$. We denote by 0 and 1 the zero and the unit of A; by $\ominus$ the operation inverse to the group-theoretic addition $\circ$. If T is an arbitrary set, then G^T will denote the T-fold Cartesian power of G, i.e. the group of functions defined over T and assuming values of G, the addition of two functions being defined by the formula (*) of [1]. If $S \subset T$ and $\nu \in G^T$, then $\nu | S$ denotes the function ν restricted to the set S.

We recall now the definition and the simplest properties of Boolean rings with an ordered basis.

A set $M \subset A$ is an *ordered basis* of A if $0, 1 \in M$, M is ordered by the relation $<$, and A is the smallest ring containing M.

It is easily proved [3]) that if M is an ordered basis of A, then every element $a \neq 0$ of A is uniquely representable as a sum:

$$a = a_1 + b_1 + a_2 + b_2 + \ldots + a_n + b_n, \tag{1}$$

where a_i, b_i, are elements of M $(i = 1, 2, \ldots, n)$, and

$$a_1 > b_1 > a_2 > b_2 > \ldots > a_n > b_n. \tag{2}$$

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[1]) Kuratowski-Mostowski, [1], p. 214. (Numbers in brackets refer to the bibliography at the end of the paper).

[2]) Mostowski-Tarski, [2], p. 70.

[3]) Mostowski-Tarski, [2], p. 71.

Formula (1) is called the *canonical representation* of a.

A simple example of a ring with an ordered basis is furnished by the ring A_H of sets which are simultaneously open and closed in a closed subset H of the Cantor's discontinuous set C.

Indeed, every element of A_H can be obtained by means of the Boolean operations $+$ and $\cdot$ from sets of the form $H \cdot \langle 0, \alpha \rangle$, where $\langle 0, \alpha \rangle$ is a closed interval the endpoint α of which lies outside C. All sets $H \cdot \langle 0, \alpha \rangle$ form thus an ordered basis of A_H. Note that every finite and every denumerable Boolean ring is isomorphic to a ring A_H, where H is a closed subset of C homeomorphic to Stone's representative space of the ring [4]).

Theorem. If M is an ordered basis of A, then the group Γ is isomorphic to $G^{M'}$, where $M' = M - \{0, 1\}$.

Proof. From the fact that the group-operations in Γ and in $G^{M'}$ are identical it follows easily that the correspondence

$$\nu \to \nu' = \nu | M' \tag{3}$$

determines a homomorphic mapping of Γ into $G^{M'}$.

Assume that $\nu' = \nu | M'$ is the constant function z. Since $\nu(1) = z$ by the definition of Γ and since $\nu(0) = \nu(0) \circ \nu(0) = z$, we infer that $\nu(b) = z$ for each element b of the basis M. Observe now that if $a_i > b_i$, then $(a_i + b_i) + b_i = a_i$ and $(a_i + b_i) \cdot b_i = 0$, whence $\nu(a_i + b_i) \circ \nu(b_i) = \nu(a_i)$ and therefore $\nu(a_i + b_i) = \nu(a_i) \ominus \nu(b_i)$. Observe further that if the conditions (2) are satisfied, then $(a_i + b_i) \cdot (a_j + b_j) = 0$. Taking both these observations together we obtain the formula

$$\nu(a) = [\nu(a_1) \ominus \nu(b_1)] \circ [\nu(a_2) \ominus \nu(b_2)] \circ \ldots \circ [\nu(a_n) \ominus \nu(b_n)],$$

which proves that if ν vanishes for the elements of M, it vanishes identically.

Hence the kernel of the homomorphism (3) contains only the constant function z and (3) defines an isomorphic mapping of Γ into $G^{M'}$.

It remains now to prove that every $\mu \in G^{M'}$ has the form ν' for a suitable $\nu \in \Gamma$. To show this we assume that $\mu \in G^{M'}$ and we put $\mu(1) = \mu(0) = z$, and

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[4]) Horn-Tarski, [3], p. 484.

(4) $$\nu(a)=[\mu(a_1)\ominus\mu(b_1)]\circ\ldots\circ[\mu(a_n)\ominus\mu(b_n)]$$

for every element defined by the formula (1).

It is evident that if $a\in M$, then $\nu(a)=\mu(a)$, because (1) takes then the form $a=a+0$. Hence $\mu=\nu|M'$ and it remains to prove that $\nu\in\Gamma$. The condition $\nu(0)=z$ being obviously satisfied, we must only show that

$$\nu(a+\bar{a})=\nu(a)\circ\nu(\bar{a}) \quad \text{for} \quad a\cdot\bar{a}=0.$$

Let us assume that the formulae (1) and (2) hold and so do the two similar formulae for the element $\bar{a}$:

(5) $$\bar{a}=\bar{a}_1+\bar{b}_1+\bar{a}_2+\bar{b}_2+\ldots+\bar{a}_m+\bar{b}_m,$$

(6) $$\bar{a}_1>\bar{b}_1>\bar{a}_2>\bar{b}_2>\ldots>\bar{a}_m>\bar{b}_m,$$

where $\bar{a}_j$ and $\bar{b}_j$ are elements of M for $j=1,2,\ldots,m$.

We shall show that if $a\cdot\bar{a}=0$, then no $\bar{a}_j$ and no $\bar{b}_j$ lies between a_i and b_i. Indeed, assuming that $a_i>\bar{a}_j>b_i$, we have either $\bar{a}_j>\bar{b}_j\geqslant b_i$ or $\bar{a}_j>b_i>\bar{b}_j$. In the first case $\bar{a}_j+\bar{b}_j\leqslant a_i+b_i$, and hence $a\cdot\bar{a}$ contains the element $\bar{a}_j+\bar{b}_j$ which is different from 0. In the second case $\bar{a}_j+b_i\leqslant\bar{a}_j+\bar{b}_j$ and $\bar{a}_j+b_i\leqslant a_i+b_i$, and hence $a\cdot\bar{a}$ contains the element $\bar{a}_j+b_i$ which is different from 0. We have thus proved that if $a\cdot\bar{a}=0$, no $\bar{a}_j$ can lie between a_i and b_i. The proof that no $\bar{b}_j$ lies between a_i and b_i is similar.

It follows from what we have just proved that if we arrange the elements $a_1,b_1,\ldots,a_n,b_n,\bar{a}_1,\bar{b}_1,\ldots,\bar{a}_m,\bar{b}_m$ in decreasing order of magnitude, we shall obtain a sequence of the form

(7) $$a_1>b_1>\ldots>a_i>b_i\geqslant\bar{a}_1>\bar{b}_1>\ldots>\bar{a}_j>\bar{b}_j\geqslant a_{i+1}>b_{i+1}>\ldots$$

Let us first assume that no two elements of this sequence are identical. Hence the canonical representation of $a+\bar{a}$ is

$$a_1+b_1+\ldots+a_i+b_i+\bar{a}_1+\bar{b}_1+\ldots+\bar{a}_j+\bar{b}_j+a_{i+1}+b_{i+1}+\ldots,$$

and we obtain according to the formula (4)

$$\nu(a+\bar{a})=[\mu(a_1)\ominus\mu(b_1)]\circ\ldots\circ[\mu(a_i)\ominus\mu(b_i)]\circ$$
$$\circ[\mu(\bar{a}_1)\ominus\mu(\bar{b}_1)]\circ\ldots\circ[\mu(\bar{a}_j)\ominus\mu(\bar{b}_j)]\circ\ldots=$$
$$=\{[\mu(a_1)\ominus\mu(b_1)]\circ\ldots\circ[\mu(a_n)\ominus\mu(b_n)]\}\circ$$
$$\circ\{[\mu(\bar{a}_1)\ominus\mu(\bar{b}_1)]\circ\ldots\circ[\mu(\bar{a}_m)\ominus\mu(\bar{b}_m)]\}=\nu(a)\circ\nu(\bar{a}).$$

The same result holds if there are pairs of equal elements in the sequence (7). Suppose e.g. that $b_i=\bar{a}_1$. We have then

$$a+\bar{a}=a_1+b_1+\ldots+a_i+b_i+\bar{a}_1+\bar{b}_1+\ldots=a_1+b_1+\ldots+a_i+\bar{b}_1+\ldots,$$

since $b_i+\bar{a}_1=0$. Hence the canonical representation of $a+\bar{a}$ is

$$a+\bar{a}=a_1+b_1+\ldots+a_i+\bar{b}_1+\ldots,$$

and we obtain from the formula (4)

(8) $$\nu(a+\bar{a})=[\mu(a_1)\ominus\mu(b_1)]\circ\ldots\circ[\mu(a_i)\ominus\mu(\bar{b}_1)]\circ\ldots$$

On the other hand

$$\nu(a)=[\mu(a_1)\ominus\mu(b_1)]\circ\ldots\circ[\mu(a_i)\ominus\mu(b_i)]\circ\ldots,$$
$$\nu(\bar{a})=[\mu(\bar{a}_1)\ominus\mu(\bar{b}_1)]\circ\ldots\circ[\mu(\bar{a}_j)\ominus\mu(\bar{b}_j)]\circ\ldots$$

If we add the formulae for $\nu(a)$ and $\nu(\bar{a})$, $\mu(\bar{a}_1)$ and $\mu(b_i)$ will cancel, and we obtain thesame value as on the right hand- side of (8). Hence $\nu(a+\bar{a})=\nu(a)\circ\nu(\bar{a})$ in all cases and the proof of the theorem is complete.

BIBLIOGRAPHY

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