HOMOMORPHISMS, MAPPINGS AND RETRACTS

BY

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In my earlier paper [1] I examined the problem: Under what condition every homomorphism h of a given Boolean algebra $\boldsymbol{A}$ into any Boolean algebra $\boldsymbol{B}$ is induced by a point mapping. The three following cases were there considered [1]):

(1) $\boldsymbol{A}$ and $\boldsymbol{B}$ are (finitely additive) fields of sets;

(2) $\boldsymbol{A}$ and $\boldsymbol{B}$ are σ-fields of sets;

(3) $\boldsymbol{A}$ is a σ-field of sets and $\boldsymbol{B}$ is a σ-quotient algebra, i. e. $\boldsymbol{B}=\boldsymbol{X}/\boldsymbol{I}$ where $\boldsymbol{X}$ and $\boldsymbol{I}$ are respectively a σ-field and a σ-ideal of sets.

The subject of the present paper is the study of the remaining case [2]):

(4) $\boldsymbol{A}$ is a (finitely additive) field $\boldsymbol{Y}$ of subsets of a set $\mathfrak{Y}$, and $\boldsymbol{B}$ is a quotient algebra, i. e. $\boldsymbol{B}=\boldsymbol{X}/\boldsymbol{I}$ where $\boldsymbol{X}$ and $\boldsymbol{I}$ are respectively a (finitely additive) field and an ideal of subsets of a set $\mathfrak{X}$.

The main result (theorems 4.1—3) is a complete characterization of fields $\boldsymbol{Y}$ with the property

(P) Every homomorphism h of $\boldsymbol{Y}$ into any quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ is induced by a point mapping φ.

This characterization is topological. The essential notion is here Borsuk's [3]) definition of a retract and of an absolute retract.

The final §§ 6 and 7 contain some applications of the main result to the case (3), and a generalization of the concept of retract.

Terminology and notation. A mapping h of a Boolean algebra $\boldsymbol{A}$ into another Boolean algebra $\boldsymbol{B}$ is said to be a *homomorphism* if $h(A+B)=h(A)+h(B)$ and $h(A')=h(A)'$ for all $A,B\in\boldsymbol{A}$

[1]) In the case (2) and (3), homomorphisms h are supposed to be σ-additive.

[2]) The case where $\boldsymbol{A}$ is a quotient or σ-quotient algebra is not interesting since it can be reduced to the case (1) and (4), or (2) and (3) respectively. See 1.1 and Sikorski [1], p. 19.

[3]) Borsuk [1], p. 153 and p. 159.

($A+B$ and A' denote always the Boolean operations corresponding to the addition and complementation of sets).

A one-one homomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$ is called an *isomorphism*. If it exists, $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isomorphic*.

A non-empty class $\boldsymbol{X}$ of subsets of a set $\mathfrak{X}$ is said to be a *field* if it is a Boolean algebra with respect to the usual set-theoretical operations, that is, if $X,X_1\in\boldsymbol{X}$ implies $X+X_1\in\boldsymbol{X}$ and $X'=\mathfrak{X}-X\in\boldsymbol{X}$.

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two fields of sets. An isomorphism g of $\boldsymbol{X}$ onto $\boldsymbol{Y}$ is called a *total isomorphism* [4]) if it can be extended to an isomorphism between the least totally additive fields containing $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. If it exists, then $\boldsymbol{X}$ and $\boldsymbol{Y}$ are said to be *totally isomorphic*.

An *ideal* $\boldsymbol{I}$ of a field of sets is a class such that:

1° $0\neq\boldsymbol{I}\subset\boldsymbol{X}$;

2° if $X_1,X_2\in\boldsymbol{I}$, then $X_1+X_2\in\boldsymbol{I}$;

3° if $X_2\subset X_1\in\boldsymbol{I}$ and $X_2\in\boldsymbol{X}$, then $X_2\in\boldsymbol{I}$.

For any $X\in\boldsymbol{X}$ the symbol $[X]$ will denote the class of all $X_1\in\boldsymbol{X}$ such that $(X-X_1)+(X_1-X)\in\boldsymbol{I}$. The collection of all (mutually disjoint) classes $[X]$ forms a Boolean algebra denoted by $\boldsymbol{X}/\boldsymbol{I}$ and called a *quotient algebra*. The Boolean operations in $\boldsymbol{X}/\boldsymbol{I}$ are defined by the formulae:

$$[X_1]+[X_2]=[X_1+X_2],\quad [X]'=[\mathfrak{X}-X].$$

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be fields of subsets of sets $\mathfrak{X}$ and $\mathfrak{Y}$ respectively, and let $\boldsymbol{I}$ and $\boldsymbol{J}$ be ideals of $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. We say that a homomorphism h of $\boldsymbol{Y}/\boldsymbol{J}$ into $\boldsymbol{X}/\boldsymbol{I}$ is *induced* by a point mapping φ of $\mathfrak{X}$ into $\mathfrak{Y}$ if

$$\varphi^{-1}(Y)\in\boldsymbol{X}\quad\text{and}\quad h([Y])=[\varphi^{-1}(Y)]\quad\text{for every}\quad Y\in\boldsymbol{Y}.$$

In particular, a homomorphism h of $\boldsymbol{Y}$ into $\boldsymbol{X}/\boldsymbol{I}$ is induced by a mapping φ of $\mathfrak{X}$ into $\mathfrak{Y}$ if

$$\varphi^{-1}(Y)\in\boldsymbol{X}\quad\text{and}\quad h(Y)=[\varphi^{-1}(Y)]\quad\text{for every}\quad Y\in\boldsymbol{Y};$$

and a homomorphism h of $\boldsymbol{Y}$ into $\boldsymbol{X}$ is induced by φ if

$$\varphi^{-1}(Y)\in\boldsymbol{X}\quad\text{and}\quad h(Y)=\varphi^{-1}(Y)\quad\text{for every}\quad Y\in\boldsymbol{Y}.$$

A topological space $\mathfrak{S}$ is *totally disconnected* if for every pair $s_1,s_2\in\mathfrak{S}$ $(s_1\neq s_2)$ there is a both open and closed set H such

[4]) This notion is due to Marczewski [2], p. 136.

that $s_1 \epsilon H$ and $s_2 \epsilon \mathfrak{S} - H$. A totally disconnected bicompact space will be termed a *B-space*.

A (closed) subset $\mathfrak{S}$ of a topological space $\mathfrak{S}_0$ is a *retract* of $\mathfrak{S}_0$ if there exists a continuous mapping $\varkappa$ of $\mathfrak{S}_0$ onto $\mathfrak{S}$ such that $\varkappa(s) = s$ for every $s \epsilon \mathfrak{S}$. The mapping $\varkappa$ is called a *retract mapping* $(\mathfrak{S}, \mathfrak{S}_0)$.

A topological space $\mathfrak{S}$ is said to be an *absolute B-retract* provided $\mathfrak{S}$ is a retract of every B-space $\mathfrak{S}_0$, $\mathfrak{S} \subset \mathfrak{S}_0$. An absolute B-retract is also a B-space.

1. Lemmas. In this section $\boldsymbol{X}, \boldsymbol{Y}$, and $\boldsymbol{Z}$ are fields of subsets of sets $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ respectively. $\boldsymbol{I}$ and $\boldsymbol{J}$ are ideals of $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively.

It follows directly from the definition that

1.1 *A mapping φ induces a homomorphism h of $\boldsymbol{Y}/\boldsymbol{J}$ into $\boldsymbol{X}/\boldsymbol{I}$ if and only if φ induces the homomorphism g (of $\boldsymbol{Y}$ into $\boldsymbol{X}/\boldsymbol{I}$) defined by the formula*

$$g(Y) = h([Y]) \epsilon \boldsymbol{X}/\boldsymbol{I} \quad \text{for} \quad Y \epsilon \boldsymbol{Y}.$$

The following lemma is obvious:

1.2 *If a mapping φ of $\mathfrak{X}$ into $\mathfrak{Y}$ induces a homomorphism h of $\boldsymbol{Y}/\boldsymbol{J}$* (or: *of $\boldsymbol{Y}$*) *into $\boldsymbol{X}/\boldsymbol{I}$, and if a mapping ψ of $\mathfrak{Y}$ in $\mathfrak{Z}$ induces a homomorphism g of $\boldsymbol{Z}$ into $\boldsymbol{Y}/\boldsymbol{J}$* (or: *into $\boldsymbol{Y}$*), *the mapping $\psi\varphi$ of $\mathfrak{X}$ into $\mathfrak{Z}$ induces the homomorphism hg of $\boldsymbol{Z}$ into $\boldsymbol{X}/\boldsymbol{I}$.*

The property (P) is not invariant under isomorphisms. However,

1.3 *The property* (P) *is invariant under total isomorphisms* (i.e. if $\boldsymbol{Y}$ has the property (P), and $\boldsymbol{Z}$ is totally isomorphic to $\boldsymbol{Y}$, then $\boldsymbol{Z}$ has also the property (P)).

This follows from 1.2 and the following lemma which is an easy consequence of a theorem of Marczewski[5]):

1.4 *An isomorphism g of $\boldsymbol{Z}$ onto $\boldsymbol{Y}$ is a total isomorphism if and only if both g and g^{-1} are induced by point mappings.*

2. *B*-spaces. For every topological space $\mathfrak{S}$, the symbol $\boldsymbol{K}(\mathfrak{S})$ will denote the field of all both open and closed subsets of $\mathfrak{S}$. By 1.3 and 1.4 the property (P) of $\boldsymbol{K}(\mathfrak{S})$ is a topological invariant of the space $\mathfrak{S}$.

[5]) See Marczewski [2], p. 140, (ii).

Stone[6]) has proved that

2.1 *Every Boolean algebra (in particular, every field of sets) $\boldsymbol{A}$ is isomorphic to the field $\boldsymbol{K}(\mathfrak{S})$ of a B-space $\mathfrak{S}$.*

If $\mathfrak{S}_1$ is another B-space such that $\boldsymbol{K}(\mathfrak{S}_1)$ is isomorphic to $\boldsymbol{A}$, then $\mathfrak{S}$ and $\mathfrak{S}_1$ are homeomorphic. Stone's space $\mathfrak{S}$ is thus uniquely determined by $\boldsymbol{A}$. It will be denoted by $\mathfrak{S}_{\boldsymbol{A}}$.

If a space $\mathfrak{S}$ is bicompact, every two-valued measure on $\boldsymbol{K}(\mathfrak{S})$ is trivial. Consequently[7]),

2.2 *If $\mathfrak{S}$ is a B-space, every homomorphism of $\boldsymbol{K}(\mathfrak{S})$ into any field of sets $\boldsymbol{X}$ is induced by a point mapping φ.*

Suppose $\mathfrak{S}_0$ is a B-space and $\mathfrak{S}$ is a closed subset of $\mathfrak{S}_0$. Then $\mathfrak{S}$ is also a B-space. Let $\boldsymbol{J}$ be the class of all sets $T \epsilon \boldsymbol{K}(\mathfrak{S}_0)$ with $S_0 \mathfrak{S} = 0$. $\boldsymbol{J}$ is an ideal of $\boldsymbol{K}(\mathfrak{S}_0)$. For every $S \epsilon \boldsymbol{K}(\mathfrak{S})$ there is a set $T \epsilon \boldsymbol{K}(\mathfrak{S}_0)$ such that $S = \mathfrak{S} S_0$. If $S_1 \epsilon \boldsymbol{K}(\mathfrak{S}_0)$ be another set with $S = \mathfrak{S} S_1$, then $(S_1 - S_0) + (S_0 - S_1) \epsilon \boldsymbol{J}$. Consequently the formula

$$g(S) = [S_0], \quad \text{where} \quad S \epsilon \boldsymbol{K}(\mathfrak{S}), \quad S_0 \epsilon \boldsymbol{K}(\mathfrak{S}_0), \quad \text{and} \quad S = \mathfrak{S} S_0,$$

defines an isomorphism of $\boldsymbol{K}(\mathfrak{S})$ onto $\boldsymbol{K}(\mathfrak{S}_0)/\boldsymbol{J}$, called *the natural isomorphism* $(\mathfrak{S}, \mathfrak{S}_0)$.

2.3 *A closed subset $\mathfrak{S}$ of a B-space $\mathfrak{S}_0$ is a retract of $\mathfrak{S}_0$ if and only if the natural isomorphism $(\mathfrak{S}, \mathfrak{S}_0)$ is induced by a mapping $\varkappa$ of $\mathfrak{S}_0$ into $\mathfrak{S}$.*

More precisely:

A mapping $\varkappa$ is a retract mapping $(\mathfrak{S}, \mathfrak{S}_0)$ if and only if $\varkappa$ induces the natural isomorphism $(\mathfrak{S}, \mathfrak{S}_0)$.

This follows from the fact that a mapping $\varkappa$ of $\mathfrak{S}_0$ into $\mathfrak{S}$ is a retract mapping $(\mathfrak{S}, \mathfrak{S}_0)$ if and only if

$$\varkappa^{-1}(S) \epsilon \boldsymbol{K}(\mathfrak{S}_0) \quad \text{and} \quad S = \mathfrak{S} \varkappa^{-1}(S) \quad \text{for every} \quad S \epsilon \boldsymbol{K}(\mathfrak{S}),$$

i. e. if $g(S) = [\varkappa^{-1}(S)] \epsilon \boldsymbol{K}(\mathfrak{S}_0)/\boldsymbol{J}$, q. e. d.

3. The space $\mathfrak{C}_\mu$. Let $\mathfrak{C}_\mu$ denote the set of all sequences $c = \{c_\xi\}_{\xi < \omega_\mu}$ where $c_\xi = 0$ or 1. Let $\mathfrak{C}_\mu^\xi$ be the set of all sequences $c \epsilon \mathfrak{C}_\mu$ such that $c_\xi = 1$, and let $\boldsymbol{C}_\mu$ be the least field (of subsets of $\mathfrak{C}_\mu$) containing all sets $\mathfrak{C}_\mu^\xi$. Consider $\mathfrak{C}_\mu$ as a topological space

[6]) Stone [1], p. 378.
[7]) See my paper [1], pp. 9-10.

with $\boldsymbol{C}_\mu$ as the class of neighbourhoods. The so-defined space $\mathfrak{C}_\mu$ is a *B*-space and $\boldsymbol{C}_\mu=\boldsymbol{K}(\mathfrak{C}_\mu)$.

Every closed subset of $\mathfrak{C}_\mu$ is also a *B*-space. Conversely:

3.1 *If $\mathfrak{S}$ is a B-space and $\overline{\overline{\boldsymbol{K}(\mathfrak{S})}}<\aleph_\mu$, then $\mathfrak{S}$ is homeomorphic to a closed subset of $\mathfrak{C}_\mu$.*

The homeomorphism is given by the characteristic function [8]) of a (transfinite) sequence $\{S_\xi\}_{\xi<\omega_\mu}$ which contains all sets $S\epsilon\boldsymbol{K}(\mathfrak{S})$.

3.2 *Let $\boldsymbol{J}$ be an ideal of $\boldsymbol{C}_\mu$. Every homomorphism h of $\boldsymbol{C}_\mu/\boldsymbol{J}$ into any quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ is induced by a point mapping.*

On account of 1.1 it is sufficient te prove that

3.3 *The field $\boldsymbol{C}_\mu$ has the property* (P).

Let h be a homomorphism of $\boldsymbol{C}_\mu$ into a quotient algebra $\boldsymbol{X}/\boldsymbol{I}$, and let $X_\xi\epsilon\boldsymbol{X}$ be a set such that $h(\mathfrak{C}_\mu^\xi)=[X_\xi]$. The characteristic function φ of the sequence $\{X_\xi\}_{\xi<\omega_\mu}$ satisfies the equation $\varphi^{-1}(\mathfrak{C}_\mu^\xi)=X_\xi$. Consequently

$$h(\mathfrak{C}_\mu^\xi)=[\varphi^{-1}(\mathfrak{C}_\mu^\xi)].$$

Hence, by induction,

$$h(C)=[\varphi^{-1}(C)]\quad\text{for every}\quad C\epsilon\boldsymbol{C}_\mu,$$

i. e. the mapping φ induces the homomorphism h.

4. Fundamental theorems on the inducing of homomorphisms. We shall now prove the following theorem:

4.1 *The three following conditions are equivalent for a B-space $\mathfrak{S}$:*
(i) *$\boldsymbol{K}(\mathfrak{S})$ has the property* (P);
(ii) *$\mathfrak{S}$ is an absolute B-retract;*
(iii) *$\mathfrak{S}$ is homeomorphic to a retract of a space $\mathfrak{C}_\mu$.*

(i)→(ii). Suppose $\mathfrak{S}\subset\mathfrak{S}_0$ where $\mathfrak{S}_0$ is a *B*-space. By (i), the natural isomorphism $(\mathfrak{S},\mathfrak{S}_0)$ is induced by a point mapping. Consequently, on account of 2.3, $\mathfrak{S}$ is a retract of $\mathfrak{S}_0$.

(ii)→(iii) follows from 3.1.

(iii)→(i). Since the property (P) of $\boldsymbol{K}(\mathfrak{S})$ is a topological invariant, it is sufficient to prove this implication in the case where $\mathfrak{S}$ is a retract of $\mathfrak{C}_\mu$. Let h be a homomorphism of $\boldsymbol{K}(\mathfrak{S})$ into a quotient algebra $\boldsymbol{X}/\boldsymbol{I}$, let $\boldsymbol{J}$ be the ideal of all sets $C\epsilon\boldsymbol{C}_\mu$ such that $\mathfrak{S}C=0$, and let g be the natural isomorphism $(\mathfrak{S},\mathfrak{C}_\mu)$. By 3.2, the

[8]) That is, a mapping $\varphi(s)=\{c_\xi\}\epsilon\mathfrak{C}_\mu$ such that $c_\xi=1$ if $s\epsilon S_\xi$ and $c_\xi=0$ if $s\epsilon\mathfrak{S}-S_\xi$. See Stone [2], p. 29 and 32, and Marczewski [1], p. 211.

homomorphism hg^{-1} of $\boldsymbol{C}_\mu/\boldsymbol{J}$ in $\boldsymbol{X}/\boldsymbol{I}$ is induced by a mapping ψ. By 2.3, the isomorphism g is induced by a retract mapping $(\mathfrak{S},\mathfrak{C}_\mu)\varkappa$. By 1.2, the homomorphism $h=(hg^{-1})g$ is induced by the mapping $\varphi=\varkappa\psi$.

4.2 *A field $\boldsymbol{Y}$ has the property* (P) *if and only if it is totally isomorphic to the field $\boldsymbol{K}(\mathfrak{S})$, where $\mathfrak{S}$ is an absolute B-retract.*

More precisely (see 2.1):

4.3 *In order that a field $\boldsymbol{Y}$ have the property* (P) *it is necessary and sufficient that $\boldsymbol{Y}$ be totally isomorphic to $\boldsymbol{K}(\mathfrak{S}_{\boldsymbol{Y}})$ and that Stone's space $\mathfrak{S}_{\boldsymbol{Y}}$ be an absolute B-retract.*

The sufficiency follows from 1.3 and 4.1. Suppose $\boldsymbol{Y}$ has the property (P) and let g be an isomorphism of $\boldsymbol{Y}$ onto $\boldsymbol{K}(\mathfrak{S}_{\boldsymbol{Y}})$. Thus g is induced by a point mapping. By 2.2 the converse isomorphism g^{-1} is also induced by a point mapping. Consequently, by 1.4, $\boldsymbol{Y}$ and $\boldsymbol{K}(\mathfrak{S}_{\boldsymbol{Y}})$ are totally isomorphic. On account of 1.3 and 4.1, the space $\mathfrak{S}_{\boldsymbol{Y}}$ is an absolute *B*-retract.

Consider the following property of a class $\boldsymbol{L}$ of sets:

(S) every subclass $\boldsymbol{L}_0\subset\boldsymbol{L}$ of mutually disjoint sets is at most enumerable.

4.4 *If a field $\boldsymbol{Y}$ has the property* (P), *it has also the property* (S).

By theorem 4.3, the space $\mathfrak{S}_{\boldsymbol{Y}}$ is then a continuous image of $\mathfrak{C}_\mu$. The property (S) is invariant under continuous mappings and the field $\boldsymbol{C}_\mu$ has this property [9]). Consequently $\boldsymbol{K}(\mathfrak{S}_{\boldsymbol{Y}})$ hat the property (S). Since $\boldsymbol{Y}$ is isomorphic to $\boldsymbol{K}(\mathfrak{S}_{\boldsymbol{Y}})$, the field $\boldsymbol{Y}$ has also this property, q. e. d.

It follows from 4.4 that, for instance, if $\boldsymbol{Y}$ is the field of all subsets of a non-enumerable set or the field of all Borel subsets of a non-enumerable space, then the field $\boldsymbol{K}(\mathfrak{S}_{\boldsymbol{Y}})$ has not the property (P).

The property (S) is not a sufficient condition for the field $\boldsymbol{K}(\mathfrak{S}_{\boldsymbol{Y}})$ to have the property (P) [10]).

[9]) See Marczewski [3], p. 131, (vi) and p. 142, (i).

[10]) Let $\mathfrak{S}$ be the *B*-space obtained by splitting every point of the segment $\langle 0,1\rangle$ into two new points. Then $\boldsymbol{K}(\mathfrak{S})$ has the property (S). Since $\mathfrak{S}$ is not a continuous image of a space $\mathfrak{C}_\mu$, the field $\boldsymbol{K}(\mathfrak{S})$ *has not* the property (P). See Šanin [1].

5. The representation problem of von Neumann and Stone. J. v. Neumann and M. H. Stone [1] have examined the question under what condition a quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ has the following property:

(R) $\boldsymbol{X}$ contains a subfield $\boldsymbol{X}_0$ such that, for every $X \epsilon \boldsymbol{X}$, there is exactly one set $X_0 \epsilon \boldsymbol{X}_0$ with $[X]=[X_0]$.

5.1 *A quotient algebra $\boldsymbol{A}=\boldsymbol{X}/\boldsymbol{I}$ has the property* (R) *if and only if, for every B-space $\mathfrak{S}$, every homomorphism h of $\boldsymbol{K}(\mathfrak{S})$ into $\boldsymbol{X}/\boldsymbol{I}$ is induced by a point mapping φ.*

Suppose $\boldsymbol{X}/\boldsymbol{I}$ has the property (R). Then the formula

$$g(S)=X_0, \quad \text{where} \quad X_0 \epsilon \boldsymbol{X}_0, \quad [X_0]=h(S), \quad S \epsilon \boldsymbol{K}(\mathfrak{S}),$$

defines a homomorphism g of $\boldsymbol{K}(\mathfrak{S})$ into $\boldsymbol{X}_0$ induced by a mapping φ on account of 2.2. We have

$$h(S)=[\varphi^{-1}(S)] \quad \text{for every} \quad S \epsilon \boldsymbol{K}(\mathfrak{S}),$$

that is, φ induces h.

On the other hand, if Stone's isomorphism g of $\boldsymbol{K}(\mathfrak{S}_{\boldsymbol{A}})$ onto $\boldsymbol{A}$ is induced by a mapping φ, then the class $\boldsymbol{X}_0$ of all sets $\varphi^{-1}(S)$ where $S \epsilon \boldsymbol{K}(\mathfrak{S}_{\boldsymbol{A}})$ satisfies the condition (R).

6. σ-homomorphisms. $\boldsymbol{X}/\boldsymbol{I}$ is called a *σ-quotient algebra* if $\boldsymbol{X}$ is a σ-field [11]) of sets, and $\boldsymbol{I}$ is a σ-ideal [12]).

For every B-space $\mathfrak{S}$, the symbol $\boldsymbol{K}_\sigma(\mathfrak{S})$ denotes the least σ-field containing the field $\boldsymbol{K}(\mathfrak{S})$.

Analogously as lemma 1.3 one can prove that

6.1 *The following property of a σ-field $\boldsymbol{Y}$ is invariant under total isomorphisms*:

(P_σ) every σ-homomorphism (i. e. a σ-additive homomorphism) of $\boldsymbol{Y}$ into any σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ is induced by a point mapping φ.

6.2 *If $\mathfrak{S}$ is an absolute B-retract, every σ-homomorphism h of $\boldsymbol{K}_\sigma(\mathfrak{S})$ into any σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ is induced by a mapping φ.*

On account of 4.1 there is a mapping φ such that

$$h(S)=[\varphi^{-1}(S)] \quad \text{for every} \quad S \epsilon \boldsymbol{K}(\mathfrak{S}).$$

h being a σ-homomorphism, the last formula holds also for every $S \epsilon \boldsymbol{K}_\sigma(\mathfrak{S})$, q. e. d.

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[11]) A field $\boldsymbol{X}$ is called a *σ-field* if $X_n \epsilon \boldsymbol{X}$ $(n=1,2,\ldots)$ implies $X_1+X_2+\ldots \epsilon \boldsymbol{X}$.

[12]) An ideal $\boldsymbol{I}$ is called a *σ-ideal* if $X_n \epsilon \boldsymbol{X}$ $(n=1,2,\ldots)$ implies $X_1+X_2+\ldots \epsilon \boldsymbol{I}$.

The field of all Borel subsets of a topological space $\mathfrak{T}$ will be denoted by $\boldsymbol{B}(\mathfrak{T})$. $\mathfrak{T}$ is said to be a *Borel space* provided it is homeomorphic to a Borel subset of the Hilbert cube.

6.3 *If $\mathfrak{T}$ is a Borel space, every σ-homomorphism h of $\boldsymbol{B}(\mathfrak{T})$ in any σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ is induced by a point mapping* [13]).

If $\mathfrak{T}$ is at most enumerable, $\mathfrak{T}=(t_1,t_2,\ldots)$, let $X_1, X_2, \ldots$ be disjoint sets of $\boldsymbol{X}$ such that $h((t_n))=[X_n]$ and $\mathfrak{X}=X_1+X_2+\ldots$. Then the mapping $\varphi(x)=t_n$ for $x \epsilon X_n$ induces h.

If $\mathfrak{T}$ is non-enumerable, theorem 6.3 follows from 6.1-2 and from the fact that the field $\boldsymbol{B}(\mathfrak{T})$ is totally isomorphic [14]) to the field $\boldsymbol{B}(\mathfrak{C}_0)=\boldsymbol{K}_\sigma(\mathfrak{C}_0)$ ($\mathfrak{C}_0$ is obviously Cantor's discontinuous set).

7. Final remarks. Theorem 4.2 gives a topological characterization of fields with the property (P). It is also possible to characterize such fields without introducing the topological terminology. In fact, the study of B-spaces is the study of perfect [15]) reduced [16]) fields of sets, and conversely.

The topological notion „retract" which is fundamental in this paper can be so defined in the terms of the theory of fields:

Let $\boldsymbol{Z}$ be a field of subsets of a set $\mathfrak{Z}$, and let U be a subset of $\mathfrak{Z}$ which does belong to $\boldsymbol{Z}$ or not. The class of all sets UZ where $Z \epsilon \boldsymbol{Z}$ will be denoted by $U\boldsymbol{Z}$. It is a field of subsets of U.

The field $U\boldsymbol{Z}$ is said to be a *retract* of $\boldsymbol{Z}$ if there is a mapping $\varkappa$ of $\mathfrak{Z}$ into U (called *retract mapping* $(U, \boldsymbol{Z})$) such that [17])

$$\varkappa^{-1}(Z) \epsilon \boldsymbol{Z} \quad \text{and} \quad U\varkappa^{-1}(Z)=Z \quad \text{for every} \quad Z \epsilon U\boldsymbol{Z}.$$

A field $\boldsymbol{Y}$ is called an *absolute retract* if, for every field $\boldsymbol{Z}$, every field $U\boldsymbol{Z}$ totally isomorphic to $\boldsymbol{Y}$ is a retract of $\boldsymbol{Z}$.

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[13]) This theorem was proved in another way in my paper [1] (theorem 4.3, p. 17).

[14]) For there is a generalized homeomorphism (in the sense of Kuratowski) of $\mathfrak{S}$ onto $\mathfrak{S}_0$. See Kuratowski [1], p. 280 and p. 358.

[15]) A field $\boldsymbol{Y}$ is *perfect* if every two-valued measure on $\boldsymbol{Y}$ is trivial. See Sikorski [1], p. 9.

[16]) A field $\boldsymbol{Y}$ of sets is *reduced* if for every pair $y_1 \neq y_2$ there is a set $Y \epsilon \boldsymbol{Y}$ such that $y_1 \epsilon Y$ and y_2 non ϵ Y.

[17]) If $U\boldsymbol{Z}$ is reduced, then $\varkappa$ has the characteristic property: $\varkappa(z)=z$ for every $z \epsilon U$.

The natural isomorphism $(U, \boldsymbol{Z})$ is obviously the isomorphism

$$g(Z) = [Z_0] \epsilon \boldsymbol{Z}/\boldsymbol{J} \quad \text{for} \quad Z \epsilon U\boldsymbol{Z},$$

where $Z_0 \epsilon \boldsymbol{Z}$, $Z = UZ_0$, and $\boldsymbol{J}$ is the ideal of all $Z \epsilon \boldsymbol{Z}$ such that $UZ = 0$.

The following theorem analogous to 2.3 follows immediately from the definition:

7.1 *A mapping* $\varkappa$ *of* $\mathfrak{Z}$ *into* U *is a retract mapping* $(U, \boldsymbol{Z})$ *if and only if it induces the natural isomorphism* $(U, \boldsymbol{Z})$.

The following theorem is a generalization of 3.1:

7.2 *If* $\overline{\overline{\boldsymbol{Y}}} \leqslant \aleph_\mu$, *there is a set* $U \subset \mathfrak{C}_\mu$ *such that the field* $\boldsymbol{Y}$ *is totally isomorphic to* UC_μ.

The following theorem corresponds to 4.1:

7.3 *The three following conditions are equivalent for any field* $\boldsymbol{Y}$:

(i) $\boldsymbol{Y}$ *has the property* (P);

(ii) $\boldsymbol{Y}$ *is an absolute retract*;

(iii) $\boldsymbol{Y}$ *is totally isomorphic to a retract of a field* $\boldsymbol{C}_\mu$.

It must be remarked that theorems analogous to 7.1-3 hold also for σ-fields. In order to obtain these theorems we must only replace the words: "field", "quotient algebra", "the property (P)" by the words: "σ-field", "σ-quotient algebra", "the property (P_σ)", etc. Instead of $\boldsymbol{C}_\mu = \boldsymbol{K}(\mathfrak{C}_\mu)$ we must consider the σ-field $\boldsymbol{K}_\sigma(\mathfrak{C}_\mu)$. It follows from a theorem of Sierpiński[18]) that every class $\boldsymbol{L} \subset \boldsymbol{K}_\sigma(\mathfrak{C}_\mu)$ of mutually disjoint sets is of potency $\leqslant 2^{\aleph_0}$. Consequently, if a σ-field $\boldsymbol{Y}$ has the property (P_σ) every class $\boldsymbol{K} \subset \boldsymbol{Y}$ of mutually disjoint sets is of potency $\leqslant 2^{\aleph_0}$ (see the analogous theorem 4.4).

REFERENCES

K. Borsuk [1], *Sur les rétractes*, Fundamenta Mathematicae 17 (1931), pp. 152-170.

C. Kuratowski [1], *Topologie I*, Monografie Matematyczne, Warszawa-Wrocław 1948.

E. Marczewski (Szpilrajn) [1], *The characteristic function of a sequence of sets and some of its applications*, Fundamenta Mathematicae 31 (1938), pp. 207-223.

— [2], *On the isomorphism and the equivalence of classes and sequences of sets*, Fundamenta Mathematicae 32 (1939), pp. 133-148.

[18]) Sierpiński [1], p. 200, Théorème 2.

— [3], *Séparabilité et multiplication cartésienne des espaces topologiques*, Fundamenta Mathematicae 34 (1947), pp. 127-143.

J. von Neumann and M. H. Stone [1], *The determination of representative elements in the residual classes of a Boolean algebra*, Fundamenta Mathematicae 25 (1935), pp. 353-378.

W. Sierpiński [1], *Un théorème sur les familles de fonctions et son application aux espaces topologiques (Solution d'un problème de R. Sikorski)*, Colloquium Mathematicum 2 (1951), pp. 198-201.

R. Sikorski [1], *On the inducing of homomorphisms by mappings*, Fundamenta Mathematicae 36 (1949), pp. 7-22.

M. H. Stone [1], *Applications of the theory of Boolean rings to general topology*, Transactions of the American Mathematical Society 41 (1937), pp. 375-481.

— [2], *On characteristic functions of families of sets*, Fundamenta Mathematicae 33 (1945), pp. 27-33.

Н. А. Шанин [1], *О произведении топологических пространств*, Труды Математического Института имени В. А. Стеклова 24, Москва-Ленинград 1948.

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