HOMOMOFORMS, MAPPINGS AND RETRACTS

by

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In my earlier paper [1] I examined the problem: Under what condition every homomorphism $h$ of a given Boolean algebra $A$ into any Boolean algebra $B$ is induced by a point mapping. The three following cases were there considered:

1. $A$ and $B$ are (finitely additive) fields of sets;
2. $A$ and $B$ are $o$-fields of sets;
3. $A$ is a $o$-field of sets and $B$ is a quotient algebra, i.e. $B=X/I$ where $X$ and $I$ are respectively a $o$-field and a $o$-ideal of sets.

The subject of the present paper is the study of the remaining case:

4. $A$ is a (finitely additive) field $Y$ of subsets of a set $\mathcal{Y}$, and $B$ is a quotient algebra, i.e. $B=X/I$ where $X$ and $I$ are respectively a (finitely additive) field and an ideal of subsets of a set $\mathcal{X}$.

The main result (theorems 4.1 - 3) is a complete characterization of fields $Y$ with the property

(?) Every homomorphism $h$ of $Y$ onto a quotient algebra $X/I$ is induced by a point mapping $\varphi$.

This characterization is topological. The essential notion is here Borsuk's** definition of a retract and of an absolute retract.

The final §§ 6 and 7 contain some applications of the main result to the case (3), and a generalization of the concept of retract.

Terminology and notation. A mapping $h$ of a Boolean algebra $A$ into another Boolean algebra $B$ is said to be a homomorphism if $h(A+B)=h(A)+h(B)$ and $h(A')=h(A')$ for all $A, B$.

1) In the case (2) and (3), homomorphisms $h$ are supposed to be $o$-additive.

2) The case in which $A$ is a quotient or $o$-quotient algebra is not interesting since it can be reduced to the case (1) and (4), or (2) and (3) respectively. See [1] and Sikorski [4], p. 199.

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(A+$B$ and $A'$ denote always the Boolean operations corresponding to the addition and complementation of sets).

A one-one homomorphism of $A$ onto $B$ is called an isomorphism. If it exists, $A$ and $B$ are said to be isomorphic.

A non-empty class $X$ of subsets of a set $\mathcal{X}$ is said to be a field if it is a Boolean algebra with respect to the usual set-theoretical operations, that is, if $X, X_1, X_2$ implies $X_1 + X_2 \in X$ and $X_1 \cdot X_2 \in X$.

Let $X$ and $Y$ be two fields of sets. An isomorphism $g$ of $X$ onto $Y$ is called a total isomorphism if it can be extended to an isomorphism between the least totally additive fields containing $X$ and $Y$ respectively. If it exists, then $X$ and $Y$ are said to be totally isomorphic.

An ideal $I$ of a field of sets is a class such that:

1. $0 \in I \subset X$;
2. if $X_1, X_2 \in I$, then $X_1 + X_2 \in I$;
3. if $X_1 \in X \in I$ and $X_1 \in X$, then $X_1 \in I$.

For any $X_1 \in X$ the symbol $[X]$ will denote the class of all $X_1 \in X$ such that $(X_1 - X) + (X_1 - X) \in I$. The collection of all (mutually disjoint) classes $[X]$ for a Boolean algebra denoted by $X/I$ and called a quotient algebra. The Boolean operations in $X/I$ are defined by the formulae:

$[X_1] + [X_2] = [X_1 + X_2]$, $[X_1] = [X_1 - X]$. Let $X$ and $Y$ be fields of subsets of sets $\mathcal{X}$ and $\mathcal{Y}$ respectively, and let $I$ and $J$ be ideals of $X$ and $Y$ respectively. We say that a homomorphism $h$ of $Y$ into $X/I$ is induced by a point mapping $\varphi$ of $X$ into $\mathcal{Y}$ if

\[ \varphi^{-1}(Y) \subseteq X \]
\[ h([Y]) = [\varphi^{-1}(Y)] \quad \text{for every } Y \in Y. \]

In particular, a homomorphism $h$ of $Y$ into $X/I$ is induced by a mapping $\varphi$ of $X$ into $\mathcal{Y}$ if

\[ \varphi^{-1}(Y) \subseteq X \]
\[ h([Y]) = [\varphi^{-1}(Y)] \quad \text{for every } Y \in Y. \]
and a homomorphism $h$ of $Y$ into $X$ is induced by $\varphi$ if

\[ \varphi^{-1}(Y) \subseteq X \]
\[ h([Y]) = [\varphi^{-1}(Y)] \quad \text{for every } Y \in Y. \]

A topological space $S$ is totally disconnected if for every pair $x, y \in S$ there is a both open and closed set $H$ such

7) This notion is due to Marczewski [2], p. 196.
that \( s \in H \) and \( s \in E \) -- \( H \). A totally disconnected bicompact space will be termed a \( B \)-space.

A (closed) subset \( E \) of a topological space \( E \) is a retract of \( E \) if there exists a continuous mapping \( x \) of \( E \) onto \( E \) such that \( x(e) = e \) for every \( e \in E \). The mapping \( x \) is called a retract mapping \((E, E)\).

A topological space \( E \) is said to be an absolute \( B \)-retract provided \( E \) is a retract of every \( B \)-space \( E \), \( E \subseteq E \). An absolute \( B \)-retract is also a \( B \)-space.

1. **Lemmas.** In this section \( X, Y, \) and \( Z \) are fields of subsets of sets \( X, Y, \) and \( Z \) respectively. \( I \) and \( J \) are ideals of \( X \) and \( Y \) respectively.

It follows directly from the definition that

1.1 A mapping \( h \) induces a homomorphism \( h \) of \( Y \) into \( X \) if and only if \( h \) induces the homomorphism \( g \) of \( Y \) into \( X \) defined by the formula

\[
g(Y) = h([Y]) = x(Y) \quad \text{for} \quad Y \subseteq X.
\]

The following lemma is obvious:

1.2 If a mapping \( g \) of \( X \) into \( Y \) induces a homomorphism \( h \) of \( Y \) into \( X \) (or \( X \) into \( Y \), and if a mapping \( f \) of \( Y \) into \( Z \) induces a homomorphism \( h \) of \( Z \) into \( Y \) (or \( Z \) into \( Y \), the mapping \( g f \) of \( X \) into \( Z \) induces the homomorphism \( h g \) of \( Z \) into \( X \).

The property \( (P) \) is not invariant under isomorphisms. However,

1.3 The property \( (P) \) is invariant under total isomorphisms (i.e. if \( Y \) has the property \( (P) \), and \( Z \) is totally isomorphic to \( Y \), then \( Z \) has also the property \( (P) \)).

This follows from 1.2 and the following lemma which is an easy consequence of a theorem of Marczewski:

1.4 An isomorphism \( g \) of \( Z \) onto \( X \) is a total isomorphism if and only if both \( g \) and \( g^{-1} \) are induced by point mappings.

2. **B-spaces.** For every topological space \( E \), the symbol \( K(E) \) will denote the field of all both open and closed subsets of \( E \). By 1.3 and 1.4 the property \( (P) \) of \( K(E) \) is a topological invariant of the space \( E \).

\(^{11}\) See Marczewski [11], p. 140, 141.

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**Stone** has proved that

2.1 Every Boolean algebra (in particular, every field of sets) \( A \) is isomorphic to the field \( K(E) \) of a \( B \)-space \( E \).

If \( E \) is another \( B \)-space such that \( K(E) \) is isomorphic to \( A \), then \( E \) and \( E \) are homeomorphic. Stone's space \( E \) is thus uniquely determined by \( A \). It will be denoted by \( E_A \).

If a space \( E \) is bicompact, every two-valued measure on \( K(E) \) is trivial. Consequently,

2.2 If \( E \) is a \( B \)-space, every homomorphism of \( K(E) \) into any field of sets \( X \) is induced by a point mapping \( 
\]

Suppose \( E \) is a \( B \)-space and \( E \) is a closed subset of \( E \). Then \( E \) is also a \( B \)-space. Let \( J \) be the class of all sets \( T \subseteq E \) with \( T \subseteq E \), \( J \) is an ideal of \( K(E) \). For every \( S \subseteq K(E) \) there is a set \( T \in K(E) \) such that \( S = [E]_T \). If \( S \subseteq K(E) \) is another set with \( S = [E]_T \), then \( S = S_j \), \( S = S_j \). Consequently the formula

\[
g(S) = [S]_T, \quad S \subseteq K(E), \quad S \subseteq K(E), \quad S = [E]_T,
\]

defines an isomorphism of \( K(E) \) onto \( K(E) \), called the natural isomorphism \((E, E)\).

2.3 A closed subset \( E \) of a \( B \)-space \( E \) is a retract of \( E \), if and only if the natural isomorphism \((E, E)\) is induced by a mapping \( x \) of \( E \) into \( E \).

More precisely:

A mapping \( x \) is a retract mapping \((E, E)\) if and only if \( x \) induces the natural isomorphism \((E, E)\).

This follows from the fact that a mapping \( x \) of \( E \) into \( E \) is a retract mapping \((E, E)\) if and only if

\[
x^{-1}(S) = [x^{-1}(S)]_T \quad \text{for every} \quad S \subseteq K(E),
\]

i.e., if \( g(S) = [x^{-1}(S)]_T \subseteq K(E), \) q. e. d.

3. **The space \( E \).** Let \( E \) denote the set of all sequences \( \epsilon = (\epsilon_n)_{n \in \mathbb{N}} \) where \( \epsilon_0 \equiv 0 \) or \( 1 \). Let \( E \) be the set of all sequences \( \epsilon \in E \), such that \( \epsilon_0 = 1 \), and let \( E \) be the least field (of subsets of \( E \)) containing all sets \( \mathbb{N}^+ \). Consider \( E \) as a topological space.

\(^{11}\) Stone [4], p. 278.

\(^{12}\) See my paper [1], pp. 9-10.
with $C_\gamma$ as the class of neighbourhoods. The so-defined space $C_\gamma$ is a B-space and $C_\gamma' = K(C_\gamma)$.

Every closed subset of $C_\gamma$ is also a B-space. Conversely:

3.1 If $\mathfrak{S}$ is a B-space and $K(\mathfrak{S}) \subseteq \mathfrak{S}$, then $\mathfrak{S}$ is homeomorphic to a closed subset of $C_\gamma$.

The homeomorphism is given by the characteristic function $^*\eta$ of a (transfinite) sequence $\{S_1\}_{n=1}^\infty$ which contains all sets $S \in K(\mathfrak{S})$.

3.2 Let $J$ be an ideal of $C_\gamma$. Every homomorphism $h$ of $C_\gamma/J$ into any quotient algebra $X/I$ is induced by a point mapping.

On account of 3.1 it is sufficient to prove that

3.3 The field $C_\gamma$ has the property (P).

Let $h$ be a homomorphism of $C_\gamma$ into a quotient algebra $X/I$, and let $X \subseteq X$ be a set such that $h(\mathfrak{c}) = [x]_I$. The characteristic function $\phi$ of the sequence $\{X_\eta\}_{\eta < \alpha}$ satisfies the equation $\phi^{-1}(\mathfrak{c}) = X_\eta$. Consequently

$$h(\mathfrak{c}) = [\phi^{-1}(\mathfrak{c})]_I.$$

Hence, by induction,

$$h(\mathfrak{c}) = [\phi^{-1}(\mathfrak{c})]_I$$

for every $\mathfrak{c} \subseteq C_\gamma$, i.e. the mapping $\phi$ induces the homomorphism $h$.

4. Fundamental theorems on the inducing of homomorphisms. We shall now prove the following theorems:

4.1 The three following conditions are equivalent for a B-space $\mathfrak{S}$:

(i) $K(\mathfrak{S})$ has the property (P);

(ii) $\mathfrak{S}$ is an absolute B-retract;

(iii) $\mathfrak{S}$ is homeomorphic to a retract of a space $C_\gamma$.

(i) $\Rightarrow$ (ii) Suppose $\mathfrak{S} \subset C_\gamma$, where $\mathfrak{S}$ is a B-space. By (ii), the natural isomorphism $(\mathfrak{S}, \mathfrak{S})$ is induced by a point mapping. Consequently, by account of 2.3, $\mathfrak{S}$ is a retract of $C_\gamma$.

(iii) $\Rightarrow$ (i) Since the property (P) of $K(\mathfrak{S})$ is a topological invariant, it is sufficient to prove this implication in the case where $\mathfrak{S}$ is a retract of $C_\gamma$. Let $h$ be a homomorphism of $K(\mathfrak{S})$ into a quotient algebra $X/I$, let $J$ be the ideal of all sets $C_\gamma C_\gamma'$ such that $S \subseteq C_\gamma'$, and let $g$ be the natural isomorphism $(\mathfrak{S}, \mathfrak{S})$. By 3.3, the homomorphism $hg^{-1}$ of $C_\gamma/J$ in $X/I$ is induced by a mapping $\phi$.

By 2.3, the isomorphism $g$ is induced by a retract mapping $(\mathfrak{S}, \mathfrak{S})$. By 1.2, the homomorphism $h = hg^{-1} g$ is induced by the mapping $\phi = \phi'$.  

4.2 A field $\mathfrak{Y}$ has the property (P) if and only if it is totally isomorphic to the field $K(\mathfrak{S})$, where $\mathfrak{S}$ is an absolute B-retract.

More precisely (see 2.1):

4.3 In order that a field $\mathfrak{Y}$ has the property (P) it is necessary and sufficient that $\mathfrak{Y}$ be totally isomorphic to $K(\mathfrak{S})$ and that Stone's space $\mathfrak{S}$ be an absolute B-retract.

The sufficiency follows from 1.3 and 4.1. Suppose $\mathfrak{Y}$ has the property (P) and let $g$ be an isomorphism of $\mathfrak{Y}$ onto $K(\mathfrak{S})$. Thus $g$ is induced by a point mapping. By 2.3, the converse isomorphism $g^{-1}$ is induced by a point mapping. Consequently, by 1.4, $\mathfrak{Y}$ and $K(\mathfrak{S})$ are totally isomorphic. On account of 1.3 and 4.1, the space $\mathfrak{S}$ is an absolute B-retract.

Consider the following property of a class $L$ of sets:

(S) every subclass $L' \subseteq L$ of mutually disjoint sets is at most enumerable.

4.4 If a field $\mathfrak{Y}$ has the property (P), it has also the property (S).

By theorem 4.3, the space $\mathfrak{S}$ is then a continuous image of $C_\gamma$. The property (S) is invariant under continuous mappings and the field $C_\gamma$ has this property. Consequently $K(\mathfrak{S})$ has the property (S). Since $\mathfrak{S}$ is isomorphic to $K(\mathfrak{S})$, the field $\mathfrak{Y}$ also has this property, q. e. d.

It follows from 4.4 that, for instance, if $\mathfrak{Y}$ is the field of all subsets of a non-enumerable set or the field of all Borel subsets of a non-enumerable space, then the field $K(\mathfrak{S})$ has not the property (P).

The property (S) is not a sufficient condition for the field $K(\mathfrak{S})$ to have the property (P).

1) See Marczewski [5], p. 151, (vi) and p. 142, (iv).

2) Let $\mathfrak{S}$ be the B-space obtained by splitting every point of the segment $(0,1)$ into two new points. Then $K(\mathfrak{S})$ has the property (S). Since $\mathfrak{S}$ is not a continuous image of a space $C_\gamma$, the field $K(\mathfrak{S})$ has not the property (P). See Śniit [1].
5. The representation problem of von Neumann and Stone. J. v. Neumann and M. H. Stone [1] have examined the question under what condition a quotient algebra \(A/I\) has the following property:

(R) \(X\) contains a subfield \(X_0\) such that for every \(X \subseteq X_0\), there is exactly one set \(X_0 \subseteq X_0\) with \([X]:=[X_0]\).

5.1 A quotient algebra \(A/I\) has the property (R) if and only if, for every \(B\)-space \(\mathcal{E}\), every homomorphism \(h\) of \(K(\mathcal{E})\) into \(X/I\) is induced by a point mapping \(\varphi\).

Suppose \(X/I\) has the property (R). Then the formula

\[ g(S) = \varphi, \quad \text{where} \quad [X_0] = h(S), \quad S \in K(\mathcal{E}), \]

defines a homomorphism \(g\) of \(K(\mathcal{E})\) into \(X_0\) induced by a mapping \(\varphi\) on account of 3.2. We have

\[ h(S) = [\varphi^{-1}(S)] \quad \text{for every} \quad S \in K(\mathcal{E}), \]

that is, \(g\) induces \(h\).

On the other hand, if Stone's isomorphism \(g\) of \(K(\mathcal{E})_0\) onto \(A\) is induced by a mapping \(\varphi\), then the class \(X_0\) of all sets \(\varphi^{-1}(S)\) where \(S \in K(\mathcal{E})_0\) satisfies the condition (R).

6. \(\sigma\)-homomorphisms. \(X/I\) is called a \(\sigma\)-quotient algebra if \(X\) is a \(\sigma\)-field \(2)\) of sets, and \(I\) is a \(\sigma\)-ideal \(2)\).

For every \(B\)-space \(\mathcal{E}\), the symbol \(K(\mathcal{E})_0\) denotes the least \(\sigma\)-field containing the field \(K(\mathcal{E})\).

Analogously as lemma 3.1 one can prove that

6.1 The following property of a \(\sigma\)-field \(Y\) is invariant under total isomorphisms:

\(L\) each \(\sigma\)-homomorphism (i.e. a \(\sigma\)-additive homomorphism) of \(Y\) into any \(\sigma\)-quotient algebra \(X/I\) is induced by a point mapping \(\varphi\).

6.2 If \(\mathcal{E}\) is an absolute \(\beta\)-retract, every \(\sigma\)-homomorphism \(h\) of \(K(\mathcal{E})_0\) into any \(\sigma\)-quotient algebra \(X/I\) is induced by a mapping \(\varphi\).

On account of 4.1 there is a mapping \(\varphi\) such that

\[ h(S) = [\varphi^{-1}(S)] \quad \text{for every} \quad S \in K(\mathcal{E}). \]

\(h\) being a \(\sigma\)-homomorphism, the last formula holds also for every \(S \in K(\mathcal{E})_0\), \(\sigma\). e. d.

1) A field \(X\) is called a \(\sigma\)-field if \(X_{\sigma}X\) (\(n = 1, 2, \ldots\)) implies \(X_{\sigma}X + \ldots = X\).

2) An ideal \(I\) is called a \(\sigma\)-ideal if \(X_{\sigma}X\) (\(n = 1, 2, \ldots\)) implies \(X_{\sigma}X + \ldots = I\).

The field of all Borel subsets of a topological space \(\mathcal{X}\) will be denoted by \(B(\mathcal{X})\). \(\mathcal{X}\) is said to be a Borel space provided it is homeomorphic to a Borel subset of the Hilbert cube.

6.3 If \(\mathcal{X}\) is a Borel space, every \(\sigma\)-homomorphism \(h\) of \(B(\mathcal{X})\) in any \(\sigma\)-quotient algebra \(X/I\) is induced by a point mapping \(\varphi\).

If \(\mathcal{X}\) is at most enumerable, \(\mathcal{X} = \{t_1, t_2, \ldots\}\), let \(X_1, X_2, \ldots\) be disjoint sets of \(X\) such that \(h(t_i) = [X_i]\) and \(\mathcal{X} = X_1 + X_2 + \ldots\). Then the mapping \(\varphi(x) = t_i\) for \(x \in X_i\), induces \(h\).

If \(\mathcal{X}\) is non-enumerable, theorem 6.3 follows from 6.1-2 and from the fact that the field \(B(\mathcal{X})\) is totally isomorphic \(1)\) to the field \(B(\mathcal{E}_0) = K(\mathcal{E}_0)\) (\(\mathcal{E}_0\) is obviously Cantor's discontinuous set).

7. Final remarks. Theorem 4.2 gives a topological characterization of fields with the property (F). It is also possible to characterize such fields without introducing the topological terminology. In fact, the study of \(B\)-spaces is the study of perfect \(1)\) reduced \(1)\) fields of sets, and conversely.

The topological notion "retract" which is fundamental in this paper can be so defined in the terms of the theory of fields:

Let \(Z\) be a field of subsets of a set \(\mathcal{Z}\), and let \(U\) be a subset of \(\mathcal{Z}\) which does belong to \(Z\) or not. The class of all sets \(UC\), where \(ZeU\) will be denoted by \(UC\). It is a field of subsets of \(U\).

The field \(UC\) is said to be a retract of \(Z\) if there is a mapping \(x \rightarrow UC\) into \(U\) (called retract mapping \(U, Z\)) such that

\[ x 
\[ U \quad \text{and} \quad U \rightarrow (Z,U) \quad \text{for every} \quad ZeU. \]

A field \(Y\) is called an absolute retract if, for every field \(Z\), every field \(UZ\) totally isomorphic to \(Y\) is a retract of \(Z\).

(1) This theorem was proved in another way in my paper [1] (theorem 4.3, p. 17).

(2) For there is a generalized homomorphism (in the sense of Kurošowski) of \(Z\) onto \(Z\). See Kuratowski [1], p. 290 and p. 595.

(3) A field \(Z\) is perfect if every two-valued measure on \(Z\) is trivial. See Sikorski [1], p. 9.

(4) A field \(Y\) of sets is reduced if for every pair \(y_0, y_1\), there is a set \(Y_{0, 1}\) such that \(y_0, y_1 \subseteq Y_{0, 1}\) and \(y_0 \neq y_1\).

(5) If \(UZ\) is reduced, then \(x\) has the characteristic property: \(x(y) = x\) for every \(y \in U\).

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The natural isomorphism \((U, Z)\) is obviously the isomorphism

\[ g(Z) = [Z_0] \in ZJ \text{ for } Z \in ZJ, \]

where \(Z_0 \in Z\), \(Z = UZ_0\), and \(J\) is the ideal of all \(Z \in Z\) such that 

\[ UZ = 0. \]

The following theorem analogous to 2.3 follows immediately from the definition:

7.1 A mapping \(x \in X\) into \(U\) is a retract mapping \((U, Z)\) if and only if it induces the natural isomorphism \((U, Z)\).

The following theorem is a generalization of 3.1:

7.2 If \(Y \subseteq K\), there is a set \(U \subseteq C\) such that the field \(Y\) is totally isomorphic to \(UC\).

The following theorem corresponds to 4.1:

7.3 The three following conditions are equivalent for any field \(Y\):

(i) \(Y\) has the property \((P)\);

(ii) \(Y\) is an absolute retract;

(iii) \(Y\) is totally isomorphic to a retract of a field \(C\).

It must be remarked that theorems analogous to 7.1-3 hold also for \(\sigma\)-fields. In order to obtain these theorems we must only replace the words: "field", "field", "the property \((P)\)" by the words: "\(\sigma\)-field", "\(\sigma\)-quotient algebra", "the property \((P)\)\(\sigma\)", etc. Instead of \(C = K(\mathbb{C})\) we must consider the \(\sigma\)-field \(K(\mathbb{C})\).

It follows from a theorem of Sierpiński [1] that every class \(L \subseteq K(\mathbb{C})\) of mutually disjoint sets is of potency \(\leq 2^\kappa\). Consequently, if a \(\sigma\)-field \(Y\) has the property \((P)\) every class \(K \subseteq Y\) of mutually disjoint sets is of potency \(\leq 2^\kappa\) (see the analogous theorem 4.4).

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* Sierpiński [1], p. 200, Théorème 2.