Now if $A \subset B$, let $C = B - A$. As we have just shown, $A \circ C = A + C = B$ and $A \circ B = A \circ A \circ C = C = B - A$.

In any case $A \circ B = A \dot{-} B$. A similar argument shows that $B \circ A = A \dot{-} B$.

4⁰ *If A is any set, then $A^{-1} = A$.*

It is sufficient to show that $A^{-1} \subset A$. Suppose not. Choose $p \epsilon A^{-1} - A$. Then $A \circ (p) = A + (p)$ and $A^{-1} - (p) = A^{-1} \circ (p) = [(p) \circ A]^{-1} = [A + (p)]^{-1}$. Furthermore we have $[A + (p)]^2 = [A \circ (p)] \circ [(p) \circ A] = A^2$.

Now for any set B we have $B = B^{-1} \circ B^2 \subset B^{-1} + B^2$. In this relation put $B = A + (p)$. We obtain

$$A + (p) \subset [A + (p)]^{-1} + [A + (p)]^2 = [A^{-1} - (p)] + A^2.$$

Since $A^2 \subset A$ and $p \, non \, \epsilon A$, we find that p appears in the set on the left of this relation but not in the set on the right: a contradiction.

5⁰ *If A and B are disjoint, then $A \circ B = A + B = B \circ A$.*

$B = A \circ A \circ B \subset A + (A \circ B)$, hence $B \subset A \circ B$. Similarly, $A \subset A \circ B$. Hence $A + B \subset A \circ B \subset A + B$. In the same way $B \circ A = A + B$.

6⁰ *If A and B are any two sets, then $A \circ B = A \dot{-} B$.*

$$A \circ B = [(A - B) \circ AB] \circ [AB \circ (B - A)] = (A - B) \circ (B - A) = (A - B) + (B - A) = A \dot{-} B.$$

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ON A PROBLEM OF SIKORSKI

BY

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Sikorski has posed the following problem [1]):

For each $\gamma < \omega_\mu$ let Z_γ be a family of sequences composed of zeros and ones, each of ordinal type γ; and suppose, for any $\delta < \gamma$, Z_δ consists exactly of those sequences which are segments of type δ of the sequences of Z_γ. Does it follow that there is a family Z of sequences of type ω_μ, such that every Z_γ consists of the segments of type γ of the sequences belonging to Z?

The answer is evidently positive if ω_μ is the limit of a sequence of ordinals of type ω, which implies that μ is a limit ordinal.

We shall show that the answer is negative for every ω_μ smaller than the first regular initial ordinal whose index is a limit ordinal, unless ω_μ is the limit of a denumerable sequence; and that the answer is negative whenever μ is not a limit ordinal.

The proof consists in constructing a counter-example.

First suppose μ is not a limit ordinal, and write $\mu = \tau + 1$. Consider the sequences $A'_\beta = \{\varrho_\gamma\}_{\gamma < \beta}$ of type β (for an arbitrary $\beta < \omega_\mu$), composed of non-zero ordinals smaller than ω_τ, such that no ordinal appears twice in A'_β, and such that the ordinals smaller than ω_τ which do *not* appear form themselves a sequence of type ω_τ. For each $\beta < \omega_\mu$ denote the set of all such sequences by Z'_β. Then for any $\alpha < \beta$, Z'_α consists exactly of the segments of type α of the sequences of Z'_β.

For each sequence A'_β of the sort defined we shall construct a sequence $A_{\nu(\beta)} = \{a_\gamma\}_{\gamma < \nu(\beta)}$ of zeros and ones, of type $\nu(\beta)$, where for each $\alpha \leqslant \beta$ we set $\nu(\alpha) = \sum_{\gamma < \alpha} \varrho_\gamma$. Take $a_0 = 1$, and adjoin a sequence of zeros of type $(-1 + \varrho_0)$, where $(-1 + \varrho_0)$ is the unique ordinal such that $1 + (-1 + \varrho_0) = \varrho_0$, thus defining a_γ for $\gamma < \varrho_0$, all zero except the first. Continue by adjoining a sequence of

[1]) See Colloquium Mathematicum 1 (1948), p. 35, P 19.

type ϱ_1, all of whose elements are zero except the first, which is one. If a_γ has been defined for all $\gamma<\nu(\alpha)$ by means of the ϱ_γ for $\gamma<\alpha$, then add a sequence of type ϱ_α all of whose elements are zero except the first, which is one. This inductive process defines $A_{\nu(\beta)}$. We have $\beta\leqslant\nu(\beta)\leqslant\omega_\tau\beta$, so $\nu(\beta)=\beta$ whenever β is a multiple of $\omega_\tau{}^\omega$. For ordinals β of this form, define Z_β as the family of all sequences A_β derived from elements A'_β of Z'_β; the correspondence between Z_β and Z'_β is one-one for each such β. Define Z_α in general to be the family of segments of type α of sequences belonging to Z_β, for any Z_β already defined where $\alpha<\beta$.

The Z_α so constructed satisfy the conditions of the problem, but there is *no* sequence of zeros and ones of type ω_μ, each of whose segments belongs to a Z_α. For if there were such a sequence A, we could reverse the process by which the Z_α were defined to produce a sequence A' of ordinals smaller than ω_τ, in which no ordinal appears twice, and of ordinal type ω_μ.

To consider the case where μ is a limit ordinal, find λ so that ω_λ is the smallest ordinal such that ω_μ is the limit of a sequence $\{\delta_\gamma\}_{\gamma<\omega_\lambda}$ of type ω_λ. If $\lambda=0$, ω_μ is the limit of a denumerable sequence, and the solution is positive. Suppose λ is an infinite limit ordinal; then ω_λ is not the limit of any sequence of type smaller than itself. That is to say that ω_λ is a regular initial ordinal with limit index, or that $\aleph_\lambda$ is an *inaccessible cardinal.*

Certainly ω_μ is not smaller than ω_λ; *the problem remains open for this case.*

The only remaining possibility is that $\lambda>0$ is not a limit ordinal, and we can extend the preceding construction to establish a negative answer as follows:

Since λ is not a limit ordinal, we can construct sets Z_α (for all $\alpha<\omega_\lambda$) so that Z does not exist. For each sequence $A_\alpha=\{a_\gamma\}_{\gamma<\alpha}$ define a sequence $C_{\delta_\alpha}=\{c_\varepsilon\}_{\varepsilon<\delta_\alpha}$ of type δ_α by setting $c_\varepsilon=0$ unless $\varepsilon=\delta_\gamma$ for some $\gamma<\alpha$; in which case set $c_\varepsilon=a_\gamma$. Let Y_α $(\alpha<\omega_\mu)$ be the set of sequences of type α so constructed if α is some δ_γ; otherwise define the sequences of Y_α by means of the segments of sequences already defined.

The sets Y_α evidently furnish the counter-example.

SUR UN PROBLÈME DE SIKORSKI

PAR

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Dans une conférence tenue à Zurich, Sikorski a posé le problème suivant. Soient: ω_μ un nombre initial régulier[1]) et, pour chaque $\alpha<\omega_\mu$, D_α un ensemble de suites de type α formées de 0 et de 1, les D_α jouissant des propriétés suivantes:

(1) D_1 n'est pas vide,

(2) Si $\alpha<\beta<\omega_\mu$, toute suite de D_β est un prolongement[2]) d'une suite de D_α,

(3) Si $\alpha<\beta<\omega_\mu$, toute suite de D_α admet un prolongement dans D_β,

(4) $\overline{\overline{D}}_\alpha<\aleph_\mu$ $(\alpha<\omega_\mu)$.

Sous ces hypothèses, *existe-t-il une suite de type ω_μ qui soit pour tout $\alpha<\omega_\mu$ prolongement d'une suite de D_α?*

Sikorski a déjà posé le même problème[3]) pour des ensembles jouissant des propriétés (1), (2), (3), et Helson a montré[4]) que la réponse est négative pour $\omega_\mu=\omega_{\nu+1}$. En m'inspirant de sa méthode, j'ai réussi à prouver que la réponse au problème embrassant la propiété (4) est *négative pour $\omega_\mu=\omega_1$*. Nous avons ensuite remarqué, Sikorski et moi, que *sous l'hypothèse $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ il en est de même pour $\omega_{\nu+1}$, quand ω_ν est un nombre initial régulier*; cette communication est consacrée à démontrer cela par la construction d'un exemple.

Considérons des suites formées non pas de 0 et de 1, mais — ce qui revient au même d'après Helson[4]) — d'éléments appartenant à un ensemble E_ν de puissance $\aleph_\nu$. Admettons de plus que l'ensemble E_ν est ordonné. Toutes les suites considérées

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[1]) F. Hausdorff, *Mengenlehre*, 3-me édition, Berlin 1935, p. 73.

[2]) Une suite $\mathfrak{b}=\{b_\xi\}_{\xi<\beta}$ est dite *prolongement* de la suite $\mathfrak{a}=\{a_\xi\}_{\xi<\alpha}$, si $a_\xi=b_\xi$ pour $\xi<\alpha$, ce que nous noterons $\mathfrak{a}\subset\mathfrak{b}$.

[3]) voir R. Sikorski, Colloquium Mathematicum 1 (1948), p. 35, P19.

[4]) Henry Helson, *On a problem of Sikorski*, ce fascicule, p. 7-8.