ON A BANACH’S PROBLEM OF INFINITE MATRICES

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All matrices considered in this paper are infinite square matrices.

A matrix $B=(b_{k,l})$ is a permutation of a matrix $A=(a_{k,l})$ if there exists a one-to-one transformation $(k,l)=(k(i,j), l(i,j))$ of the set of all pairs of positive integers into itself, such that

$$a_{k,l} = b_{k(i,j), l(i,j)}.$$

If $k(i,j)=i$ for $i=1, 2, \ldots$, i.e., if every element remains in its line, then $B$ is said to be a line permutation of $A$. Analogously, if $l(i,j)=j$ for $j=1, 2, \ldots$, i.e., if every element remains in its column, then $B$ is said to be a column permutation of $A$.

S. Banach proposed the problem 1) whether every permutation $B$ of a matrix $A$ is a superposition of a finite sequence of line permutations and of column permutations.

The purpose of this paper is to show that the answer to Banach’s problem is affirmative:

Every permutation $B$ of a matrix $A=(a_{k,l})$ is the result of a finite alternating line and column permutations.

Proof. Let $m=f(n)$ be a transformation of the set of all positive integers onto itself such that the set $f^{-1}(m)$ is infinite for every positive integer $m$.

First step: a line permutation. Let us order lexicographically all the elements of $A$, i.e., let us put $a_{k,l} < a_{k', l'}$ if $i < f(k)$, or if $i = f(k)$ and $j < f(l)$. Let $a_{k_1,l_1}$ be the first element of $A$ which is in the $f(l)$-th line of $B$, and let $k_1 = 1+j$. By induction we define an infinite sequence $(a_{k_n,l_n})$ of elements of $A$ and an infinite increasing sequence $(k_n)$ of positive integers as follows:

- $a_{k_{n+1}, l_{n+1}}$ is the first element of $A$ which is in the $(f(n)+1)$-th line of $B$ and which is different from the elements

$$a_{k_1, l_1}, \ldots, a_{k_{n-1}, l_{n-1}}, a_{k_n, l_n}, \ldots, a_{k_k, l_k}$$

(two elements of a matrix being considered as different if their pairs of indices are not the same);

- $k_{n+1}$ is the least integer greater than

$$l_1, \ldots, l_n, l_{n+1}, k_1, \ldots, k_n.$$

It follows from this definition that

$$i_n, j_n \neq i_m, k_m \quad \text{for} \quad n, m = 1, 2, \ldots,$$

$$i_n, j_n \neq i_n, j_m \quad \text{for} \quad n \neq m,$$

$$i_n, k_m \neq i_m, k_n \quad \text{for} \quad n \neq m.$$

We now define a line permutation $A'=(a'_{i,j})$ of the matrix $A$ by the formula

$$a'_{i', j'} = a_{i_n, j_n} \quad \text{if} \quad (i, j) = (i_n, j_n),$$

$$a'_{i', j'} = a_{i, j} \quad \text{for all other pairs} \quad (i, j).$$

Thus the element $a'_{i_n, k_n}$ lies in the $f(n)$-th line of $B$.

Let $(l_m)$ be a sequence of positive integers defined by the equalities:

$$l_n = i_n, \quad \text{and} \quad l_n = 1 \quad \text{for all other m}.$$  

The sequence $e=(l'_m, l_m)$ has the property that every column of $A'$ contains exactly one element of $e$ and, for every $n$, the sequence $e$ contains an enumerable number of elements lying in the $n$-th line of $B$.

Second step: a column permutation. Let $a_{k_n, l_n}$ be the subsequence of $e$ containing all elements of $e$ which lie in the $n$-th line of $B$.

The permutation $A''=(a''_{i,j})$ of $A'$ is defined by the equalities:

$$a''_{i_n, m} = a'_{l_n, m}$$

$$a''_{i, l_n} = a'_{i, l_n}$$

$$a''_{i, j} = a'_{i, j} \quad \text{for all other pairs} \quad (i, j).$$

Every set $f^{-1}(e)$ being infinite, any line of $A''$ contains an enumerable number of elements from every line of $B$.

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1) Colloquium Mathematicum 1 (1940), p. 151, P 52.
Let $d_{s,r} = (a_{i,j})_{i=1,...,n}^{s,r}$ denote the sequence containing all elements $a_{i,j}$ from the $s$-th line of $A^n$ which lie in the $r$-th line of $B$.

**Third step: a line permutation.** Let $m(s,r)$ be an infinite square matrix of positive integers such that $m(1,r) = r$ and that every positive integer appears exactly once in every line and exactly once in every column of this matrix. An example of such a matrix is the following one:

$$
\begin{align*}
m(1,r) &= r, \quad r=1,2,... \\
m(2^n+1,r) &= (r-2^n) \quad \text{for} \quad n \leq r \leq (n+1)2^n \\
m(2^n-1,r) &= m(t,m(2^n+1,r)) \quad \text{for} \quad 1 < t < 2^n, q=1,2,..., n=0,1,2,..., r=1,2,...
\end{align*}
$$

Explicitly:

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Let $A'' = (a_{i,j})_{i=1,2,...}^{n,m,n}$ be the matrix defined by the equality:

$$
a_{i,n,m,n} = a_{m(2^n-1,r)}^{n,n,m,1,n,1}.
$$

In other words, the construction of $A''$ is as follows: the first line of $A''$ is identical with that of $A'$; in the $s$-th line we put the elements of the sequence $d_{s,m(2^n-1,r)}$ under the elements of the sequence $d_{s,r}$ in the same order. Obviously, the matrix $A''$ is a line permutation of $A'$ and it has the following property:

(i) every column of $A''$ contains exactly one element from the $m$-th line of $B$ ($m=1,2,...$).

In fact, let us consider the $p$-th column of $A''$. We have $p = p_{n,m,n}$ for some $r$ and $n$. Thus

$$
a_{i,r}^{n,m,n} = a_{i,n,m(2^n-1,r)}^{n,n,m,1,n,1}.
$$

By definition of $m(2^n, r)$ there exists exactly one integer $s$ such that $m = m(s,r)$; and by definition of $d_{s,r}$ the element $a_{i,j}^{n,m,n}$ lies in the $m$-th column of $B$ if and only if $m(s,r) = m$. This proves the property (i).

**Fourth step: a column permutation.** We put each element $a_{i,j}$ of $A''$ on the place $(i,k)$, where $k$ is the number of that line of $B$ which contains the element $a_{i,j}^{n,m,n}$. By (i) we obtain in this way a matrix $A'''$ with the property:

(ii) every element lies in $A'''$ in the same line as in $B$.

**Fifth step: a line permutation.** We put each element $a_{i,j}^{n,m,n}$ of $A'''$ on the place $(j,k)$, where $j$ is the number of that column of $B$ which contains the element $a_{i,j}^{n,m,n}$. By (ii) this line permutation gives the matrix $B$.

**Remark.** The theorem proved above holds also for matrices which have a finite number of lines and an infinite number of columns. The method of the proof is the same.