AN EXAMPLE OF A FINITE DIMENSIONAL CONTINUUM HAVING AN INFINITE NUMBER OF CARTESIAN FACTORS

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A set A is called a *Cartesian factor* of the space M, if there exists a set B such that the cartesian product $A\times B$ is homeomorphic to M. Obviously, for every space M, the set consisting only of one point is a Cartesian factor of M, and the same applies to every set homeomorphic to M. If a space M has no other Cartesian factors, then it is called *prime* [1]).

Obviously, every finite dimensional continuum has a decomposition into a finite number of prime factors. The question arises whether the number of all topologically different prime Cartesian factors of such a continuum can be infinite. In this paper we shall give an affirmative answer to this question.

Theorem. There exists in the 3-dimensional Euclidean space a 3-dimensional absolute retract having an infinite number of topologically different prime Cartesian factors.

Proof. For every natural n consider in the Euclidean (x,y) — plane two segments:

K_n with the end-points $\left(\frac{1}{n},0\right)$ and $\left(\frac{1}{n},-\frac{1}{n}\right)$,

L_n with the end-points $\left(\frac{1}{n},0\right)$ and $\left(\frac{1}{n}+\frac{1}{n^2},-\frac{1}{n}\right)$.

Obviously,

$$(K_n+L_n)\cdot(K_m+L_m)=0 \qquad \text{for } m\neq n.$$

Let Q denote the square $0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 1$.

Putting

$$A_n=Q+L_n+\sum_{m=1}^{\infty}K_m.$$

we obviously obtain a sequence $\{A_n\}$ of topologically different continua which are absolute retracts.

[1]) Comp. K. Borsuk, *Sur la décomposition des polyèdres en produits cartésiens*, Fundamenta Mathematicae 31 (1938), p. 137.

We shall show that all the sets

$$K_1\times A_n, \qquad n=1,2,\ldots$$

are homeomorphic. In fact, $K_1\times A_n$ consists of the cube $K_1\times Q$ and of a sequence of the rectangles $K_1\times L_n$, $K_1\times K_1$, $K_1\times K_2,\ldots$, each of which has one of its sides common with the surface S of the cube $K_1\times Q$. Let K_m' denote the side of the rectangle $K_1\times K_m$ lying on S. We see at once that $\{K_m'\}$ are disjoint and constitute a sequence convergent to an edge of the cube $K_1\times Q$. There are two rectangles $K_1\times K_n$ and $K_1\times L_n$ which adjoin to the segment K_n', and there is only one such rectangle $K_1\times K_m$ for all remaining segments K_m'. Evidently, there exists a homeomorphic mapping φ_n of the cube $K_1\times Q$ on itself such that

$$\varphi_n(K_m')=K_{m+1}' \qquad \text{for } m=1,2,\ldots,n-1,$$
$$\varphi_n(K_n')=K_1',$$
$$\varphi_n(x)=x \qquad \text{for } x\in K_m',\ m>n.$$

It is not difficult to extend the mapping φ_n on all rectangles $K_1\times L_n, K_1\times K_1, K_1\times K_2,\ldots$ in such a manner that the extended mapping φ_n^* maps homeomorphically $K_1\times K_m$ on $K_1\times K_{m+1}$, for $m=1,2,\ldots,n-1$, and $K_1\times L_n+K_1\times K_n$ on $K_1\times L_1+K_1\times K_1$ and is identical on all others rectangles.

Thus, we obtain a homeomorphic mapping φ_n^* of the set $K_1\times A_n$ on the set $K_1\times A_1$. The last set, as a product of the segment K_1 and of the 2-dimensional absolute retract A_1, is an absolute retract of the dimension 3. Clearly, it lies in the 3-dimensional Cartesian space. Hence, all sets $K_1\times A_n$ are homeomorphic to the 3-dimensional absolute retract $K_1\times A_1$. Thus the proof of the theorem is complete.

P 90. Does there exist a finite dimensional continuum having $2^{\aleph_0}$ topologically different Cartesian factors?

P 91. Does there exist a polytope having an infinite number of topologically different factors?