L'HÔPITAL'S RULE FOR VECTOR-VALUED FUNCTIONS

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Let X be a Banach space; a vector-valued function $x(t)$, i. e. a function from a closed interval $I=[a,b]$ to X, will be said to be *weakly continuous* at t_0 if, $\xi(x)$ being any linear functional over X, we have $\xi(x(t))\to\xi(x(t_0))$ as $t\to t_0$. This function will be said to be *weakly differentiable* at t_0 to x_0 if

$$\lim_{h\to 0}\xi\left(\frac{x(t_0+h)-x(t_0)}{h}-x_0\right)=0,$$

$\xi(x)$ having the same meaning as above. The element x_0 will be written $x'_w(t_0)$.

In the sequel we shall denote by $x(t), y(t)$, the functions from I to X, by $\gamma(t)$ the real-valued functions. $\operatorname{Conv}_{a<t<b} y(t)$ will denote the convex span of the set of the values of the function $y(t)$ as t ranges in the open interval (a,b); $\operatorname{Cl} E$ will denote the closure of the set E.

In this Note L'Hôpital's rule is generalized for couples of functions the first of which is a vector-valued function and the second is real-valued. As in the case of real functions we shall base our theorem on Cauchy's Mean Value Theorem:

Let the function $x(t)$ be weakly differentiable in (a,b) and weakly continuous in $[a,b]$ and let $\gamma(t)$ be differentiable in (a,b), continuous in $[a,b]$. If $\gamma'(t)\neq 0$ in (a,b), then

$$\frac{x(b)-x(a)}{\gamma(b)-\gamma(a)}\in\operatorname{Cl}\operatorname{Conv}_{a<t<b}\frac{x'_w(t)}{\gamma'(t)}.$$

To prove this suppose the contrary; then the element

$$x_0=\frac{x(b)-x(a)}{\gamma(b)-\gamma(a)}$$

has a positive distance from the convex closed set

$$X_1=\operatorname{Cl}\operatorname{Conv}_{a<t<b}\frac{x'_w(t)}{\gamma'(t)}.$$

Hence by a theorem of Eidelheit[1]) there exists a hyperplane which separates the element x_0 from the set X_1. The equation of this hyperplane being $\xi(x)=c$, we have $\xi(x)\leqslant c$ for $x\in X_1$ and $\xi(x_0)>c$, where $\xi(x)$ is a linear functional over X. This is, however, impossible since by Cauchy's Mean Value Theorem on the reals there exists a $\tau\in(a,b)$ such that $\xi(x_0)=\eta'(\tau)/\gamma'(\tau)$, where $\eta(t)=\xi(x(t))$.

Theorem. Let the function $x(t)$ be weakly differentiable in (a,b) and weakly continuous at b, and let $\gamma(t)$ be differentiable in (a,b) and continuous at b. Suppose moreover that $x(b)=0$, $\gamma(b)=0$, and that $\lim_{t\to b}\frac{x'_w(t)}{\gamma'(t)}$ exists. Then $\lim_{t\to b}\frac{x(t)}{\gamma(t)}$ exists and

$$\lim_{t\to b}\frac{x(t)}{\gamma(t)}=\lim_{t\to b}\frac{x'_w(t)}{\gamma'(t)}$$

(all limits in this theorem are meant strong).

Proof. By hypothesis we have $\gamma'(t)\neq 0$ for t sufficiently near b. Let $t_n\to b$ and $a<t_n<b$ $(n=1,2,\ldots)$. Then

$$\frac{x(t_n)}{\gamma(t_n)}\in\operatorname{Cl}\operatorname{Conv}_{t_n<t<b}\frac{x'_w(t)}{\gamma'(t)}=X_n.$$

The diameter of the set Z_n of values of the function $y(t)=\frac{x'_w(t)}{\gamma'(t)}$ in the interval $t_n<t<b$ tends by hypothesis to 0 as $n\to\infty$. Theorem results then

1° from the fact that the diameter of Z_n is equal to the diameter of X_n,

and

2° from $X_n\supset X_{n+1}$.

[1]) M. Eidelheit, *Zur Theorie der konvexen Mengen in linearen normierten Räumen*, Studia Mathematica 6 (1936), p. 104-111.