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C O M M U N I C A T I O N S

ON THE GENERALIZED LIMIT OF BOUNDED SEQUENCES

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The purpose of this note is to give a simple proof of the following well-known theorems[1]):

A. *With every bounded sequence*[2]) $\{\xi_n\}$ *a real number* $\operatorname{Lim}_{n=\infty}\xi_n$ *can be associated so that*

(a) $\operatorname{Lim}_{n=\infty}\xi_n$ *is equal to the usual limit of a subsequence of* $\{\xi_n\}$; *consequently* $\liminf_{n=\infty}\xi_n \leqslant \operatorname{Lim}_{n=\infty}\xi_n \leqslant \limsup_{n=\infty}\xi_n$;

(b) $\operatorname{Lim}_{n=\infty}(a\xi_n+b\eta_n)=a\operatorname{Lim}_{n=\infty}\xi_n+b\operatorname{Lim}_{n=\infty}\eta_n$;

(c) $\operatorname{Lim}_{n=\infty}\xi_n\cdot\eta_n=\operatorname{Lim}_{n=\infty}\xi_n\cdot\operatorname{Lim}_{n=\infty}\eta_n$.

B. *With every bounded sequence* $\{\xi_n\}$ *a real number* $\operatorname{Lim}_{n=\infty}\xi_n$ *can be associated so that*

(a′) $\operatorname{Lim}_{n=\infty}\xi_n$ *is equal to the usual limit of a subsequence of the sequence* $\left\{\frac{\xi_1+\xi_2+\ldots+\xi_n}{n}\right\}$; *consequently*

$$\liminf_{n=\infty}\xi_n \leqslant \operatorname{Lim}_{n=\infty}\xi_n \leqslant \limsup_{n=\infty}\xi_n;$$

(b) $\operatorname{Lim}_{n=\infty}(a\xi_n+b\eta_n)=a\operatorname{Lim}_{n=\infty}\xi_n+b\operatorname{Lim}_{n=\infty}\eta_n$;

(c′) $\operatorname{Lim}_{n=\infty}\xi_n=\operatorname{Lim}_{n=\infty}\xi_{n+1}$.

Let M be the set of all bounded sequences of real numbers $x=\{\xi_n\}$, let $g(x)=\sup_n|\xi_n|$, and let $f_n(x)=\xi_n$. Obviously $|f_n(x)|\leqslant g(x)$

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[1]) See S. Banach, *Théorie des opérations linéaires* (Monografie Matematyczne, Warszawa - Lwów 1932), p. 34, and S. Mazur, *O metodach sumowalności* (in Polish), Księga Pamiątkowa I Polskiego Zjazdu Matematycznego, Supplément aux Annales de la Société Polonaise de Mathématique (1929), p. 103.

[2]) $\{\xi_n\}$ and $\{\eta_n\}$ will always denote enumerable bounded sequences of real numbers.

for $x \in M$ and $n=1,2,\ldots$ The real functions f_n defined on M may be interpreted as points of the Cartesian product $P=\mathfrak{P}_{x \in M} I_x$, where I_x denotes the interval $|t| \leqslant g(x)$. The topological space P (Tychonoff's cube) being bicompact[3]), the sequence $f_n \in P$ contains a limit point $f \in P$.

Put

$$\operatorname*{Lim}_{n=\infty} \xi_n = f(x) \quad \text{for} \quad x=\{\xi_n\} \in M.$$

Obviously we have (a). More generally, if N is a finite subset of M, then there exists a subsequence $\{f_{m_n}\}$ such that[4])

(*) $$\lim_{n=\infty} f_{m_n}(x) = f(x) \quad \text{for} \quad x \in N.$$

This is a general property of limit points in Cartesian products of metric spaces, which follows immediately from the definition of neighbourhoods.

Now let $x=\{\xi_n\} \in M$ and $y=\{\eta_n\} \in M$. Put $w=\{a\xi_n+b\eta_n\}$ and $v=\{\xi_n\eta_n\}$, and let $N=(x,y,w,v)$. By (*) we obtain

$$a\,f(x)+b\,f(y) = a \lim f_{m_n}(x) + b \lim f_{m_n}(y) = a \lim \xi_{m_n} + b \lim \eta_{m_n} =$$
$$= \lim (a\xi_{m_n}+b\eta_{m_n}) = \lim f_{m_n}(w) = f(w),$$

and

$$f(x)f(y) = \lim f_{m_n}(x) \cdot \lim f_{m_n}(y) = \lim \xi_{m_n} \cdot \lim \eta_{m_n} =$$
$$= \lim \xi_{m_n} \cdot \eta_{m_n} = \lim f_{m_n}(v) = f(v),$$

which proves (b) and (c).

Theorem A is established.

In order to prove Theorem B it is sufficient to define the now generalized limit of $\{\xi_n\}$ as $\operatorname*{Lim}_{n=\infty} \dfrac{\xi_1+\ldots+\xi_n}{n}$, where „Lim" has the meaning defined above.

Note that it is impossible to define the generalized limit so that all the conditions (b), (c), (c′) are satisfied, and $\operatorname*{Lim}_{n=\infty} 1 = 1$. In fact, let $\xi_{2n}=0$ and $\xi_{2n+1}=1$. We have $\xi_n+\xi_{n+1}=1$ and

[3]) A. Tychonoff, *Über die topologische Erweiterung von Räumen*, Mathematische Annalen 102 (1930), p. 544-561.

[4]) The sign „lim" denotes the usual limit of a sequence.

$\xi_n \cdot \xi_{n+1}=0$. The hypothesis that the conditions (b), (c), (c′) hold, implies

$$\operatorname*{Lim}_{n=\infty} \xi_n + \operatorname*{Lim}_{n=\infty} \xi_{n+1} = 1, \qquad \operatorname*{Lim}_{n=\infty} \xi_n \cdot \operatorname*{Lim}_{n=\infty} \xi_{n+1} = 0,$$

and

$$\operatorname*{Lim}_{n=\infty} \xi_n = \operatorname*{Lim}_{n=\infty} \xi_{n+1},$$

which is impossible.