REMARKS ON THE MOMENT PROBLEM AND A THEOREM OF PICONE

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Lerch published in 1903 [1]) the following theorem:

If $\int\limits_0^\infty e^{-a_n t} f(t)\,dt = 0$ *for an arithmetic sequence of positive numbers* a_n, *then* $f(t)=0$ *almost everywhere in* $(0,\infty)$.

Lerch's theorem is actually known rather in the following equivalent form:

If $\int\limits_0^1 x^{a_n} f(x)\,dx = 0$ *for an arithmetic sequence of positive numbers* a_n, *then* $f(x)=0$ *almost everywhere in* $(0,1)$.

Either form of the theorem may be deduced from the other, by convenient substitution.

To prove his theorem, Lerch used Weierstrass' approximation theorem and the same method is usually given in books.

Next year, in 1904, Phragmén [2]) gave another proof which was based upon the following identity:

$$(*)\quad \lim_{x\to\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kxt} \int\limits_0^T e^{-kx\tau} f(\tau)\,d\tau = \int\limits_0^t f(\tau)\,d\tau \quad \text{for } 0 \leqslant t < T,$$

where the interval $(0,T)$ is finite or infinite and f is any (real or complex) function, such that the integrals exist.

The formula (*) is easy to establish by interchanging the order of summation and integration. Thus we get

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kxt} \int\limits_0^T e^{-kx\tau} f(\tau)\,d\tau = \int\limits_0^T \varphi(x, t-\tau) f(\tau)\,d\tau,$$

[1]) M. Lerch, *Sur un point de la théorie des fonctions génératrices d'Abel*, Acta Mathematica 27 (1903), p. 339-352.

[2]) E. Phragmén, *Sur une extension d'un théorème classique de la théorie des fonctions*, Acta Mathematica 28 (1904), p. 351-368.

where

$$\varphi(x,u) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kxu} = 1 - \exp(-e^{xu}).$$

To have (*), it suffices to remark that

$$\lim_{x\to\infty} \varphi(x,u) = \begin{cases} 1 & \text{for } u>0, \\ 0 & \text{for } u<0. \end{cases}$$

In 1939, Picone [3]) used Phragmén's identity (*) to prove the following theorem:

If K, p, q, *are constants,* $p < b-a$, *and*

$$\left| \int\limits_a^b e^{x(b-t)} f(t)\,dt \right| < K|x|^q e^{p|x|}$$

as x *increases indefinitely then* $f(t)=0$ *almost everywhere in* (a,b).

This theorem appeared to be very useful in the study of uniqueness-problems in the theory of partial differential equations [4]).

It seems to be of interest to discuss some connections between Picone's theorem and the classical forms of the moment theorem.

We shall prove Picone's theorem in another and slightly stronger form.

Theorem. If the function $f(x)$ *is integrable over a finite interval* $0 \leqslant a < x < b$ *and if*

$$\int\limits_a^b x^{a_n} f(x)\,dx = o[(a+\varepsilon)^{a_n}] \qquad \textit{for any } \varepsilon > 0,$$

where $\{a_n\}$ *is an arithmetic sequence of positive numbers, then* $f(x)=0$ *almost everywhere in* (a,b).

[3]) M. Picone, *Nuove determinazioni per gl'integrali delle equazioni lineari a derivate parziali*, Rendiconti della Accademia Nazionale dei Lincei 28 (1939), p. 339-348.

[4]) M. Picone, *Nouvelles méthodes de recherche pour la détermination des intégrales des équations linéaires aux dérivées partielles*, Annales de la Société Polonaise de Mathématique 19 (1946), p. 36-61.

It is easily seen that Lerch's theorem may be regarded as a particular case of this, where $a=0$ and $b=1$. On the other hand, Picone's theorem follows by a convenient substitution. A third, direct but interesting, corollary is the following one:

Corollary. If all the moments $\int_1^b x^n f(x)\,dx$ *are bounded, then* $f(x)=0$ *almost everywhere in* $(1,b)$.

From this Corollary the theorem of Lerch may be easily deduced. In fact, if $\int_0^1 x^n f(x)\,dx=0$ $(n=1,2,\dots)$ then also $\int_0^1 w(x)f(x)\,dx=0$ for any polynomial $w(x)$. Thus

$$\int_1^2 x^n f(x-1)\,dx=\int_0^1 (1+x)^n f(x)\,dx=0$$

and consequently $f(x-1)=0$ almost everywhere in $(1,2)$, i.e. $f(x)=0$ almost everywhere in $(0,1)$. The theorem with arbitrary arithmetic sequence $\{a_n\}$ can be obtained by convenient substitution.

To prove the Theorem, we establish the following

Lemma. If

$$(**)\qquad \lim_{n\to\infty} q^n \int_0^T e^{nt} g(t)\,dt=0 \qquad (T\ \textit{finite})$$

for any $0<q<1$, *then* $g(t)=0$ *almost everywhere in* $(0,T)$.

Proof of the Lemma. Let $g(t)=f(T-t)$. Then

$$a_k(x,t)=e^{-kx(T-t)}\int_0^T e^{kx\tau} g(\tau)\,d\tau=e^{kxt}\int_0^T e^{-kx\tau} f(\tau)\,d\tau \quad (k\ \text{positive integer}).$$

If $t<T$ is fixed and x, being an integer, increases indefinitely, then $\lim_{x\to\infty} a_k(x,t)=0$ uniformly in k and consequently

$$\lim_{x\to\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}a_k(x,t)=0.$$

By (*) we have $\int_0^t f(\tau)\,d\tau=0$ and $f(\tau)=0$ almost everywhere in $0<\tau<t$. But t may be fixed arbitrarily in $(0,T)$; thus $f(\tau)=0$ and $g(\tau)=0$ almost everywhere in $(0,T)$.

Proof of the Theorem. We obtain the Theorem from the Lemma by substituting in (**)

$$e^t=\left(\frac{x}{a}\right)^{\beta},\qquad g(t)=\left(\frac{x}{a}\right)^{\alpha} f(x)\qquad (a_n=\alpha+n\beta)$$

in the case $a>0$; for $a=0$ the proof has been given above by means of the Corollary.

It is known that Lerch's theorem was strongly extended by Müntz [5]), by replacing the arithmetic sequence $\{a_n\}$ by any indefinitely increasing sequence $\{\beta_n\}$ such that $\sum_{n=1}^{\infty}\frac{1}{\beta_n}=\infty$. It would be of interest to investigate the possible extensions of the theorem given above.

[5]) Ch. H. Müntz, *Über den Approximationssatz von Weierstrass*, Mathematische Abhandlungen H. A. Schwarz gewidmet, Berlin 1914, p. 303-312.