ON MEASURES IN CARTESIAN PRODUCTS OF BOOLEAN ALGEBRAS

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Boolean algebras will be denoted by letters $\boldsymbol{A}, \boldsymbol{B}, \ldots$, their elements by $A, B, \ldots$ The symbols A', $A+B$, AB, and $\prod_{n=1}^{\infty} B_n$ will denote Boolean operations corresponding to complementation, addition, and multiplication of sets.

A finite non-negative function μ defined on a Boolean algebra $\boldsymbol{B}$ is called a *measure* if $\mu(B_1+B_2)=\mu(B_1)+\mu(B_2)$ for every pair of disjoint elements $B_1, B_2 \epsilon \boldsymbol{B}$. A measure μ is called:

strictly positive if $\mu(B)\neq 0$ for $B\neq 0$;

normalized if $\mu(0')=1$;

σ-measure if for every decreasing sequence $B_n \epsilon \boldsymbol{B}$ such that $\prod_{n=1}^{\infty} B_n=0$ in $\boldsymbol{B}$ there is $\lim_{n=\infty} \mu(B_n)=0$.

Kappos has formulated [1]) a definition of the cartesian products $\boldsymbol{A}_1 \times \boldsymbol{A}_2$ of two Boolean algebras $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. Every pair of measures μ_1 and μ_2 defined on $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, respectively, determines uniquely a measure μ on $\boldsymbol{A}_1 \times \boldsymbol{A}_2$ called the product measure of μ_1 and μ_2. Kappos has proved [2]) that if $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are σ-complete, and if μ_1 and μ_2 are strictly positive σ-measures on $\boldsymbol{A}_1$ and on $\boldsymbol{A}_2$, respectively, then the product measure μ is a σ-measure on $\boldsymbol{A}_1 \times \boldsymbol{A}_2$.

The purpose of the present paper is to show by giving of an example that Kappos' theorem is false. This example, which is an easy modification of an example of Helson and Steinhaus [3]), proves also that Banach's theorem on extending of

[1]) D. A. Kappos, *Die Cartesischen Produkte und die Multiplikation von Massfunktionen in Booleschen Algebren*, Mathematische Annalen 120 (1947), p. 43-74.

[2]) Ibidem, Satz 12, p. 65.

[3]) See H. Helson, *Remark on measures in almost independent fields*, Studia Mathematica 10 (1948), p 182-183.

σ-measures defined on independent σ-fields of sets [4]) cannot be generalized to the case of independent Boolean algebras.

The definition of cartesian products of Boolean algebras formulated in the present paper is formally different from Kappos' definition, but the both definitions are equivalent.

§ 1. Cartesian products of Boolean algebras. A subset $\boldsymbol{B}_0$ of a Boolean algebra $\boldsymbol{B}$ is called a *subalgebra* of $\boldsymbol{B}$ if $A, B \epsilon \boldsymbol{B}_0$ implies $AB \epsilon \boldsymbol{B}_0$ and $B' \epsilon \boldsymbol{B}_0$. A subalgebra $\boldsymbol{B}_0$ of a σ-complete Boolean algebra $\boldsymbol{B}$ is called a *σ-subalgebra* of $\boldsymbol{B}$ if $B_n \epsilon \boldsymbol{B}_0$ $(n=1,2,\ldots)$ implies $\prod_{n=1}^{\infty} B_n \epsilon \boldsymbol{B}_0$.

A Boolean algebra $\boldsymbol{B}$ is said to be a *cartesian product* of two Boolean algebras $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ if $\boldsymbol{B}$ contains two subalgebras $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ such that

(a) $\boldsymbol{B}_i$ is isomorphic to $\boldsymbol{A}_i$ $(i=1$ and $2)$;

(b) the subalgebras $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are independent [5]), i. e. if $0\neq B_1 \epsilon \boldsymbol{B}_1$ and $0\neq B_2 \epsilon \boldsymbol{B}_2$, then $B_1 B_2 \neq 0$;

(c) the smallest subalgebra of $\boldsymbol{B}$ which contains both $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, is $\boldsymbol{B}$ itself.

By (c) every element $B \epsilon \boldsymbol{B}$ can be represented in the form

$$(*) \qquad B=B_1^1 B_2^1+\ldots+B_1^n B_2^n,$$

where $B_1^j \epsilon \boldsymbol{B}_1$ and $B_2^j \epsilon \boldsymbol{B}_2$ $(j=1,\ldots,n)$.

If two Boolean algebras $\boldsymbol{B}$ and $\boldsymbol{C}$ are cartesian products of Boolean algebras $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, then $\boldsymbol{B}$ is isomorphic to $\boldsymbol{C}$. In fact, let $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ be subalgebras of $\boldsymbol{C}$ satisfying the conditions (a)-(c), and let h_i $(i=1$ and $2)$ be an isomorphism of $\boldsymbol{B}_i$ on $\boldsymbol{C}_i$. By (b) and (c), the isomorphisms h_1 and h_2 can be extended [6]) to an homomorphism [7]) h of $\boldsymbol{B}$ in $\boldsymbol{C}$. Analogously the converse iso-

[4]) See S. Banach, *On measures in independent fields* (edited by S. Hartman), Studia Mathematica 10 (1948), p. 159-177, Theorem 1, and E. Marczewski, *Indépendance d'ensembles et prolongement de mesures (Résultats et problèmes)*, Colloquium Mathematicum 1 (1948), p. 122-132, Théorème II_∞.

[5]) This notion is due to Marczewski, loco citato, p. 125-126.

[6]) See R. Sikorski, *On an analogy between measures and homomorphisms*, Annales de la Société Polonaise de Mathématique 23 (1950), p. 5-6, Theorem III.

[7]) A mapping h is called a *homomorphism* if $h(A+B)=h(A)+h(B)$ and $h(A')=h(A)'$. An *isomorphism* is a one-one homomorphism.

morphisms h_1^{-1} and h_2^{-1} can be extended to a homomorphism g of $\boldsymbol{C}$ in $\boldsymbol{B}$. It is easy to show that

$$gh(B)=B \text{ and } hg(C)=C \text{ for any } B\epsilon\boldsymbol{B} \text{ and any } C\epsilon\boldsymbol{C},$$

which proves that $g=h^{-1}$, i. e. that h is an isomorphism of $\boldsymbol{B}$ on $\boldsymbol{C}$.

Kappos' cartesian product [8] $\boldsymbol{L}$ of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ is the cartesian product of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ in the sense defined above. In fact, let E_i ($i=1$ and 2) denote the greatest element (the unit) of $\boldsymbol{A}_i$. Using Kappos' notation, let $\boldsymbol{B}_1$ be the class of all $\{A_1\times E_2\}$, where $A_1\epsilon\boldsymbol{A}_1$, and analogously let $\boldsymbol{B}_2$ be the class of all $\{E_1\times A_2\}$, where $A_2\epsilon\boldsymbol{A}_2$. The subalgebras $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ of $\boldsymbol{L}$ satisfy the conditions (a)-(c).

§ 2. Product measures. For $i=1$ and 2 let μ_i^* be a normalized measure on a Boolean algebra $\boldsymbol{A}_i$, and let $\boldsymbol{B}$ be a cartesian product of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. Let $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ have the same meaning as previously. By (a), the measures μ_1^* and μ_2^* induce two isomorphic measures μ_1 and μ_2 on $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, respectively. As Kappos has shown [9], there exists a normalized measure μ on $\boldsymbol{B}$ such that

$$\mu(B_1B_2)=\mu_1(B_1)\cdot\mu_2(B_2) \quad \text{for} \quad B_1\epsilon\boldsymbol{B}_1 \quad \text{and} \quad B_2\epsilon\boldsymbol{B}_2.$$

The measure μ is called the *product measure* of μ_1 and μ_2.

§ 3. An example. The closed interval $\langle 0,1\rangle$ will be denoted by $\mathfrak{J}$. Lebesgue's linear and plain measures will be denoted by $m^{(1)}$ and $m^{(2)}$, respectively.

For $i=1$ and 2 let $\boldsymbol{A}_i$ be the σ-complete Boolean algebra of measurable subsets of $\mathfrak{J}$ modulo sets of measure $m^{(1)}$ zero. The measure $m^{(1)}$ induces a σ-measure μ_i^* on $\boldsymbol{A}_i$ which is normalized and strictly positive.

Let $\boldsymbol{F}$ be the field of all subsets X of the square $\mathfrak{J}\times\mathfrak{J}$ which can be represented in the form

$$X=X_1^1\times X_2^1+\ldots+X_1^n\times X_2^n,$$

where X_i^j ($i=1$ and 2, and $j=1,\ldots,n$) are measurable subsets of $\mathfrak{J}$. Further let $\boldsymbol{I}$ be the ideal of all sets $X\epsilon\boldsymbol{F}$ with $m^{(2)}(X)=0$. Consider the Boolean algebra $\boldsymbol{B}=\boldsymbol{F}/\boldsymbol{I}$. The element (the coset) of $\boldsymbol{B}$ determined by a set $X\epsilon\boldsymbol{F}$ will be denoted by $[X]$.

[8]) Loco citato, p. 53.
[9]) Ibidem, p. 61-64.

The Boolean algebra $\boldsymbol{B}$ is a cartesian product of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. In fact, the subalgebras $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ of all elements $[X\times\mathfrak{J}]$ or $[\mathfrak{J}\times X]$, respectively, (where X is a measurable subset of $\mathfrak{J}$) satisfy the conditions (a)-(c).

The product measure of μ_1^* and μ_2^* is the measure μ defined by the formula

$$\mu([X])=m^{(2)}(X) \quad \text{for} \quad X\epsilon\boldsymbol{F}.$$

I shall prove that *the measure μ is not a σ-measure on $\boldsymbol{B}$*.

For this purpose it is sufficient to construct a decreasing sequence $S_n\epsilon\boldsymbol{F}$ ($n=1,2,\ldots$) such that

(α) $m^{(2)}\left(\prod_{n=1}^{\infty}S_n\right)>0$;

(β) there exists no set $X_1\times X_2\epsilon\boldsymbol{F}$ such that $m^{(2)}(X_1\times X_2)>0$ and $m^{(2)}(X_1\times X_2-\prod_{n=1}^{\infty}S_n)=0$.

In fact, the sequence of elements $B_n=[S_n]$ of $\boldsymbol{B}$ is also decreasing. The condition (α) means that $\lim_{n=\infty}\mu(B_n)>0$, and the condition ($\beta$) — that $\prod_{n=1}^{\infty}B_n=0$ in the Boolean algebra $\boldsymbol{B}$ (see the formula (*)).

Let N be a nowhere dense closed subset of $\mathfrak{J}$ with $m^{(1)}(N)>0$, and let S be the set of all points (x_1,x_2) such that $x_i\epsilon\mathfrak{J}$ and $|x_1-x_2|\epsilon N$. The set S has the following properties:

(i) $m^{(2)}(S)>0$,

(ii) *If* $Y_1\times Y_2\epsilon\boldsymbol{F}$ *and* $m^{(2)}(Y_1\times Y_2)>0$, *then* $Y_1\times Y_2-S\neq 0$,

(iii) *If* $X_1\times X_2\epsilon\boldsymbol{F}$ *and* $m^{(2)}(X_1\times X_2)>0$, *then* $m^{(2)}(X_1\times X_2-S)>0$.

The property (i) follows from $m^{(1)}(N)>0$.

To prove the property (ii) suppose that $Y_1\times Y_2\subset S$ for some measurable sets Y_1 and Y_2 of positive measure $m^{(1)}$. By a theorem of Steinhaus [10]), the set N_0 of numbers $|x_1-x_2|$, where $x_i\epsilon Y_i$ ($i=1$ and 2), would contain an interval, which is impossible since N_0 is a subset of the nowhere dense set N. Thus the property (ii) is proved.

In order to prove (iii) observe that X_i ($i=1$ and 2) contains a subset Y_i with $m^{(1)}(Y_i)>0$, such that $Y_iG\neq 0$ implies $m^{(1)}(Y_iG)>0$

[10]) H. Steinhaus, *Sur les distances des points des ensembles de mesure positive*, Fundamenta Mathematicae 1 (1920), p. 93-104, Théorème VII.

for every set G open in $\mathfrak{J}$. Consequently, if H is open in $\mathfrak{J}\times\mathfrak{J}$, and $H(Y_1\times Y_2)\neq 0$, then $m^{(2)}(H(Y_1\times Y_2))>0$. The set $H=\mathfrak{J}\times\mathfrak{J}-S$ is open in $\mathfrak{J}\times\mathfrak{J}$, and $H(Y_1\times Y_2)\neq 0$ on account of (ii). Therefore

$$m^{(2)}(X_1\times X_2-S)\geqslant m^{(2)}(Y_1\times Y_2-S)=m^{(2)}(H(Y_1\times Y_2))>0,$$

i. e. S has the property (iii).

Now, the set S being closed, there exists a decreasing sequence $S_n\epsilon \boldsymbol{F}$ such that $S=\prod_{n=1}^{\infty} S_n$. The property (i) implies (α), and the property (iii) implies (β). Thus the product measure μ is not a σ-measure on $\boldsymbol{B}$, q. e. d.

§ 4. Banach's theorem on extending of σ-measures. Let $\boldsymbol{B}$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, and μ have the same meaning as in § 3, and let μ_i ($i=1$ and 2) denote the measure μ restricted to arguments $B\epsilon\boldsymbol{B}_i$. Obviously μ_i is a σ-measure on $\boldsymbol{B}_i$ since it is isomorphic to μ_i^*.

Let $\boldsymbol{D}$ be MacNeille's minimal extension [11]) of the Boolean algebra $\boldsymbol{B}\subset\boldsymbol{D}$, and let $\boldsymbol{C}$ be the smallest σ-subalgebra of $\boldsymbol{D}$ which contains $\boldsymbol{B}$. Then $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are independent σ-subalgebras of $\boldsymbol{D}$, and $\boldsymbol{C}$ is the smallest σ-subalgebra of $\boldsymbol{D}$ which contains both $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.

I shall prove that *there is no σ-measure ν on $\boldsymbol{C}$ such that*

$$\nu(B_1 B_2)=\mu_1(B_1)\cdot\mu_2(B_2) \quad \text{for} \quad B_1\epsilon\boldsymbol{B}_1 \text{ and } B_2\epsilon\boldsymbol{B}_2.$$

Suppose such a σ-measure ν exists. Obviously ν would be an extension of μ. If $B_n\epsilon\boldsymbol{B}$ are elements of a decreasing sequence such that $\prod_{n=1}^{\infty} B_n=0$ in $\boldsymbol{C}$, then $\lim_{n=\infty}\mu(B_n)=\lim_{n=\infty}\nu(B_n)=0$.

On the other hand, the condition: $\prod_{n=1}^{\infty} B_n=0$ in $\boldsymbol{C}$ is equivalent to $\prod_{n=1}^{\infty} B_n=0$ in $\boldsymbol{B}$ since all infinite Boolean operations on elements of $\boldsymbol{B}$ are the same in $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ [12]). Thus the hypothesis that ν exists implies that μ is a σ-measure on $\boldsymbol{B}$, which is impossible by § 3.

[11]) H. Mac Neille, *Partially ordered sets*, Transactions of the American Mathematical Society 42 (1937), p. 416-460; see p. 437.

The above example shows that Banach's theorem on extending of σ-measures defined on independent σ-fields of sets (mentioned in § 1) fails for independent Boolean σ-algebras.

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[12]) This is a characteristic property of minimal extensions. See R. Sikorski, *Cartesian products of Boolean algebras*, Fundamenta Mathematicae 37 (1950), p. 36, theorem 3.6, and p. 30, theorems 1.1 and 1.2.