THE SO-CALLED PETERSBURG PARADOX

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In many old manuals of the Calculus of probability the following game is given as an example which is not covered by the classical theory: The banker B tosses a coin until head appears; the game is then finished, and B pays 2^{n-1} pennies to A if there were n trials necessary to show a head. The question is to determine a fair entrance fee to be paid in advance by A to B. The classical rule which determines the fee as equal to $E(x)$, the *expectation of* x promised by B, becomes illusory for this game, as

$$(1) \quad E(x)=(1/2)1+(1/4)2+\ldots+(1/2^n)2^{n-1}+\ldots=\sum_n 1/2=\infty.$$

Feller[1]) has analyzed this paradox by going back to the principle which justifies the rule in ordinary games, where $E(x)$ is finite. The "fair" fee has the property of balancing gains and losses if the game is repeated indefinitely. To speak more exactly, the net gain of a partner after N games has to be small in comparison with N, and the probability of its being so has to approach 1 as N increases indefinitely. This principle can be satisfied by a constant fee (equal to $E(x)$) in most popular games; if, however, $E(x)$ is infinite, as in the Petersburg game, it can be satisfied only by a variable fee, a_N, for the Nth repetition of the game. The determination of a_N according to the principle quoted is an application of a sort of *weak law of great numbers.*

Thus to solve the paradox we must consider not an individual game but a sequence of games. This point of view once being adopted, we can find another solution, based on a sort of *strong law of great numbers.*

This law says that if $F(a)$ is the cumulative distribution function[2]) of the random variable x, and $x_1, x_2, \ldots$ is the sequence of values taken by x in succesive independent experiments, we can assert with probability 1 that *the sequence* $\{x_N\}$ *will have* $F(a)$ *as its distribution function*; the italicised statement means that the relative frequency of terms x_N smaller than a equals $F(a)$ for every a. In our special case the random variable x can take in every game the values

$$1, \quad 2, \quad 4, \quad \ldots, \quad 2^{n-1}, \quad \ldots$$

with respective probabilities

$$1/2, \quad 1/4, \quad 1/8, \quad \ldots, \quad 1/2^n, \quad \ldots$$

To find a sequence with the same distribution function, we write first unities alternating with void places,

$$1 \quad\quad 1 \quad\quad 1 \quad\quad 1 \quad\quad 1 \quad\quad 1 \quad\quad 1 \quad\quad 1 \quad\quad 1 \quad\quad \ldots,$$

then we fill every second of the void places with a 2,

$$1\ 2\ 1 \quad\quad 1\ 2\ 1 \quad\quad 1\ 2\ 1 \quad\quad 1\ 2\ 1 \quad\quad 1 \quad\quad \ldots,$$

then every second of the void places left with a 4,

$$1\ 2\ 1\ 4\ 1\ 2\ 1 \quad\quad 1\ 2\ 1\ 4\ 1\ 2\ 1 \quad\quad 1 \quad\quad \ldots,$$

and so on. The resulting sequence

$$(2) \quad 1,\ 2,\ 1,\ 4,\ 1,\ 2,\ 1,\ 8,\ 1,\ 2,\ 1,\ 4,\ 1,\ 2,\ 1,\ 16,\ 1,\ \ldots$$

shows 1 with the frequency 1/2, 2 with the frequency 1/4, in general 2^{n-1} with the frequency $1/2^n$. Calling a_N its Nth term and fixing a_N as the entrance fee for the Nth repetition of the Petersburg game we can predict with probability 1 that the amounts b_N paid by B to A will yield a sequence with the same distribution function as the sequence (2) of the fees a_N paid (in advance) by A to B. Such equality justifies calling the game *fair* in a new sense of the word.

When asked to estimate the (constant) fee he would like to pay for the Petersburg game, the average man names in most cases an amount less than 10 pennies. The reason is his taking into account only 20 terms of the series (1) at most, as the probability of the game extending beyond the 20th trial is less than 1/1000000; his disbelief in such extraordinary occurençes is

[1]) W. Feller, Annals of Mathematical Statistics 16 (1945), p. 301-304.

[2]) Cf. Colloquium Mathematicum 1 (1947), p. 48-49; for details: H. Steinhaus, *Sur les fonctions indépendantes (VIII)*, Studia Mathematica 11 (1950), p. 133-144.

scarcely influenced by the rich reward promised by B if such a case would really happen The same man would probably not hesitate to repeat the game indefinitely, his fees being determined by the sequence (2), because he would realize that he pays great fees very rarely, as rarely as he wins, in the long run, amounts equalling such fees.