ON A RATIO TEST OF FRINK

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In a recent paper [1]) Frink gave the following test for convergence of a series of positive terms:

If for some positive integer k

$$\varlimsup_{n\to\infty}\left(\frac{a_n}{a_{n-k}}\right)^n < e^{-k}, \tag{1}$$

then the series is convergent; if for some k

$$\left(\frac{a_n}{a_{n-k}}\right)^n \geqslant e^{-k} \quad \textit{for} \quad n > N, \tag{2}$$

then the series is divergent.

It is interesting to notice that the inequality (1) is equivalent (for each positive integer k) to the inequality

$$\varlimsup_{n\to\infty} n\left(\frac{a_n}{a_{n-k}}-1\right) < -k, \tag{3}$$

i. e. from (1) follows (3), and conversely. For $k=1$ it follows that the series satisfying Frink's test of convergence are identical with those satisfying the test of Raabe.

Indeed, suppose (1). Then

$$\varlimsup_{n\to\infty}\left(\frac{a_n}{a_{n-k}}\right)^n < e^{-s} < e^{-k}, \qquad s > k.$$

Hence

$$\left(\frac{a_n}{a_{n-k}}\right)^n < \left(1-\frac{1}{n}\right)^{ns}, \qquad n > N_1;$$

$$\frac{a_{n-k}}{a_n} > \left(\frac{n}{n-1}\right)^s = \left(1+\frac{1}{n-1}\right)^s \geqslant 1+\frac{s}{n-1}, \qquad n > N_1;$$

[1]) O. Frink, *A ratio test*, Bulletin of the American Mathematical Society 55 (1948), p. 953.

$$\frac{a_n}{a_{n-k}} \leqslant \frac{1}{1+\frac{s}{n-1}} = 1-\frac{s}{n-1}+\left(\frac{s}{n-1}\right)^2-\ldots, \qquad n > N_1;$$

$$n\left(\frac{a_n}{a_{n-k}}-1\right) \leqslant -\frac{n}{n-1}s+o(1) < -k-c, \qquad c>0, \qquad n > N_2.$$

The converse implication is still more obvious. Thus (3) may be used instead of (1) as a convergence test. It may be proved independently from Frink's reasoning and equally easily, on the usual lines of proving Raabe's test by comparing $\sum a_n$ with $\sum n^{-s}$.

As for the divergence test (2) it is easy to follow from (2) the inequality

$$n\left(\frac{a_n}{a_{n-k}}-1\right) \geqslant -k, \qquad n > N_3, \tag{4}$$

but not conversely, because (4) is equivalent to

$$\left(\frac{a_n}{a_{n-k}}\right)^n \geqslant \left(1-\frac{k}{n}\right)^n, \qquad n > N_3.$$

Therefore, if a series satisfies the divergence test of Frink, it satisfies the generalized divergence test (4) of Raabe, but not conversely.

For instance, take the harmonic series

$$k=1, \qquad a_1=1, \qquad a_n=a_{n-1}\left(1-\frac{1}{n}\right).$$

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