ON STABILITY OF NON-LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

BY

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1. We shall consider here systems of the form

$$\frac{dz_i}{dt}=\sum_{j=1}^{n}a_{ij}z_j+f_i(z_1,\ldots,z_n,\frac{dz_1}{dt},\ldots,\frac{dz_n}{dt},t),\qquad i=1,\ldots n,$$

where the a_{ij} are constant and the f_i are continuous and in some sense small.

Denoting the matrix of elements a_{ij} by A, the vector with components z_i by z, and the vector with components f_i by f we have

$$\frac{dz}{dt}=Az+f(z,\frac{dz}{dt},t). \tag{1.0}$$

The norm of a vector z is denotes by $|z|=\sum_{i=1}^{n}|z_i|$. The norm of a matrix A is defined by the formula

$$|A|=\max_{1\leqslant j\leqslant n}\sum_{i=1}^{n}|a_{ij}|.$$

We shall assume that $f(0,0,t)=0$, so that $z=0$ is a solution of (1.0), and consider the Liapounoff stability [1]) of the solution $z=0$. This problem has been considered by Bellman [2]), where hypotheses involving among other things, the existence of and restrictions on $\frac{\partial f}{\partial z_i}$ and $\frac{\partial f}{\partial w_i}$, where f is $f(z,w,t)$, are assumed. As Bellman remarks in his paper, the author communicated to him a proof which does not assume the existence of these partial derivatives. This proof we now give.

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[1]) Cf. e. g. G. D. Birkhoff, *Dynamical Systems*, New York 1927, p. 122.

[2]) R. Bellman, *On the boundedness of solutions of non-linear differential and difference equations*, Transactions of the American Mathematical Society 62 (1947), p. 347-386. Other references to related work will be found here.

We shall not consider conditional stability here. By a solution of (1.0) we shall mean throughout a vector, $z(t)$ with a continuous derivative, $z'(t)$ which satisfies (1.0).

In the first three theorems we shall assume:

(1.1) *The real parts of the characteristic roots of A are all negative.*

We shall also assume that there exist two positive constants α and β, which depend only on A, such that for small $|z|$ and $|w|$ (where w is a vector with n components)

$$|f(z,w,t)|\leqslant\alpha|z|+\beta|w|,\qquad 0\leqslant t<\infty. \tag{1.2}$$

The condition (1.2) is certainly satisfied if

$$|f(z,w,t)=o(|z|+|w|)\ \text{as}\ |z|+|w|\to 0\ \text{uniformly in}\ t\geqslant 0. \tag{1.3}$$

Theorem I. Let $z(t)$ be a solution of (1.0), *and let* (1.1) *and* (1.3) *be satisfied. Then if $|z(0)|$ and $|z'(0)|$ are sufficiently small, $|z(t)|$ and $|z'(t)|$ are uniformly bounded over $0\leqslant t<\infty$ and tend to zero as $t\to\infty$. Moreover, the bounds on $|z|$ and $|z'|$ can be taken as $J|z(0)|$, where J is a constant.*

Theorem I is a consequence of Theorem II.

Theorem II. If (1.3) *is replaced by* (1.2) *in Theorem I, where α and β are two positive constants which depend only on A, then Theorem I remains true.*

Theorem III. Let $z(t)$ be a solution of

$$\frac{dz}{dt}=Az+B(t)z+f(z,\frac{dz}{dt},t), \tag{1.4}$$

where $B(t)$ is a continuous square matrix for $0\leqslant t<\infty$, and $|B(t)|\to 0$ as $t\to\infty$. Let (1.1) *and* (1.2) *be satisfied. Then the conclusion of Theorem I remains valid.*

In case (1.1) is not satisfied, and the real parts of the characteristic roots are non-positive, there is another type of stability criterion, providing the norm of every solution of the linear system with constant coefficients

$$\frac{dy}{dt}=Ay \tag{1.5}$$

is bounded as $t\to\infty$. (This is the case, for example, if the characteristic roots of A with real part zero are all distinct.)

Theorem IV. Let the norm of every solution of (1.5) *be bounded as* $t\to\infty$. *Let there exist two positive functions* $g(t)$ *and* $h(t)$, *such that for small* $|z|$ *and* $|w|$

$$|f(z,w,t)|\leqslant g(t)|z|+h(t)|w|, \tag{1.6}$$

where $g(t)$ *is uniformly bounded over* $0\leqslant t<\infty$ *while* $h(t)\leqslant 1-a$ *over* $0\leqslant t<\infty$ *for some* $0<a<1$. *Let*

$$\int_0^\infty g(t)\,dt<\infty,\qquad \int_0^\infty h(t)\,dt<\infty. \tag{1.7}$$

Then if $z(t)$ *is a solution of* (1.0), *and if* $|z(0)|$ *and* $|z'(0)|$ *are sufficiently small,* $|z(t)|$ *and* $|z'(t)|$ *are uniformly bounded over* $0\leqslant t<\infty$. *Moreover, these bounds can be taken as* $J|z(0)|$, *where* J *is a constant.*

The conditions corresponding to (1.6) in Bellman's paper contain as a factor on the right an additional term which is $o(1)$ as $|z|+|w|\to 0$, which we do not require.

2. We turn first to the proof of Theorem II. A more explicit definition of the constants α and β will be given first. Let $Y(t)$ be the matrix solution of (1.5) which is the unit matrix for $t=0$. Thus $Y'(t)=AY(t)$. Then the hypothesis (1.1) of Theorems I, II, and III implies that there exist two positive constants λ and C, depending only on A, such that

$$|Y(t)|\leqslant Ce^{-\lambda t},\qquad t\geqslant 0. \tag{2.0}$$

The constants α and β in (1.2) need satisfy only the following requirement

$$\lambda>\frac{\alpha+\beta|A|}{1-\beta}C>0. \tag{2.1}$$

Clearly (2.1) can be satisfied by choosing α and β small enough. We see that in any case $\beta<1$. It is convenient to define

$$\sigma=\frac{\alpha+\beta|A|}{1-\beta}C. \tag{2.2}$$

Then (2.1) is

$$\lambda>\sigma>0. \tag{2.3}$$

We require now the following well-known lemma:

Let $u(t)\geqslant 0$. *Let* $G(t)\geqslant 0$ *be integrable and let*

$$u(t)\leqslant b+\int_0^t G(\tau)u(\tau)\,d\tau,\qquad t\geqslant 0, \tag{2.4}$$

where b *is a constant. Then*

$$u(t)\leqslant be^{\int_0^t G(\tau)d\tau}.$$

To prove the lemma we let

$$H(t)=\int_0^t G(\tau)u(\tau)\,d\tau.$$

Then $u(t)=H'(t)/G(t)$ and (2.4) becomes $H'(t)\leqslant bG(t)+G(t)H(t)$.

Multiplying by $e^{-\int_0^t G(\tau)d\tau}$ we obtain

$$\frac{d}{dt}\left(He^{-\int_0^t G(\tau)d\tau}\right)\leqslant bG(t)e^{-\int_0^t G(\tau)d\tau}.$$

Integrating from $t=0$ we get

$$H(t)\leqslant b\left(e^{\int_0^t G(\tau)d\tau}-1\right).$$

Since (2.4) can be written $u(t)\leqslant b+H(t)$, we see that the abowe yields the result of the lemma.

Proof of Theorem II. We have from (1,0) so long as $|z|$ and $|z'|$ are small

$$|z'(t)|\leqslant|A||z|+|f|\leqslant(|A|+\alpha)|z|+\beta|z'|.$$

Or

$$|z'(t)|\leqslant\frac{|A|+\alpha}{1-\beta}|z(t)|. \tag{2.5}$$

Thus so long as $|z(t)|$ is small, $|z'(t)|$ is small and thus we need only show now that $|z(t)|$ is small.

We have (variation of constants formula or as can be verified by direct substition into (1.0))

$$z(t)=Y(t)z(0)+\int_0^t Y(t-\tau)f\Big(z(\tau),z'(\tau),\tau\Big)\,d\tau. \tag{2.6}$$

Thus so long as $|z(t)|$ is small we have from (1.2) and (2.0)

$$|z(t)| \leqslant Ce^{-\lambda t}|z(0)| + C\int_0^t e^{-\lambda(t-\tau)}(\alpha|z| + \beta|z'|)\,d\tau.$$

Or using (2.5) and (2.2)

$$|z(t)| \leqslant Ce^{-\lambda t}|z(0)| + \sigma e^{-\lambda t}\int_0^t e^{\lambda\tau}|z(\tau)|\,d\tau.$$

Setting $|z(t)|e^{\lambda t} = u(t)$ we get

$$u(t) \leqslant C|z(0)| + \sigma\int_0^t u(\tau)\,d\tau.$$

Using the lemma we have

$$u(t) \leqslant C|z(0)|e^{\sigma t}, \qquad t \geqslant 0,$$

or

$$|z(t)| \leqslant C|z(0)|e^{-(\lambda-\sigma)t} < C|z(0)|, \qquad t > 0.$$

Thus if $|z(0)|$ is small enough, then so is $|z(t)|$ for all $t>0$, and by (2.5) so is $|z'(t)|$. This completes the proof of the Theorem II and therefore also of Theorem I.

Proof of theorem III. Clearly there exists a constant $P<\infty$ such that $|B(t)| \leqslant P$. Let

$$\max_{t \geqslant t_0}|B(t)| = \gamma.$$

For $t \geqslant t_0$ we can incorporate Bz into f with the consequence that α is replaced by $\alpha+\gamma$. Choose t_0 large enough so that

$$\lambda > \frac{\alpha+\gamma+\beta|A|}{1-\beta}C. \tag{2.7}$$

Since $|B(t)| \to 0$ and since (2.1) holds, this can be done. For $0 \leqslant t \leqslant t_0$ we have in the same way as (2.5)

$$|z'(t)| \leqslant \frac{|A|+P+\alpha}{1-\beta}|z(t)|.$$

From this

$$|z(t)| \leqslant |z(0)|\exp\left(\frac{|A|+P+\alpha}{1-\beta}t\right).$$

Thus by choosing $|z(0)|$ small enough we can make $|z(t)|$ and $|z'(t)|$ as small as we wish for $0 \leqslant t \leqslant t_0$. For $t \geqslant t_0$ we simply repeat the argument of Theorem II with α replaced by $\alpha+\gamma$ and $t=0$ replaced by $t=t_0$.

Proof of Theorem IV. We note that according to the hypothesis of Theorem IV the matrix solution $Y(t)$ of (1.5), which is the unit matrix at $t=0$, satisfies

$$|Y(t)| \leqslant C, \qquad t \geqslant 0 \tag{2.8}$$

for some C.

From (1.0) and (1.6) we have so long as $|z|$ and $|z'|$ are small

$$|z'| \leqslant |A||z| + g(t)|z| + (1-a)|z'|.$$

Or

$$|z'| \leqslant {}^1/_a(|A| + g(t))|z|. \tag{2.9}$$

In other words: so long as $|z(t)|$ is small, $|z'(t)|$ will be small.

From (2.6) we find, using (1.6) and (2.8), that so long as $|z|$ and $|z'|$ are small

$$|z(t)| \leqslant C|z(0)| + C\int_0^t (g(\tau)|z(\tau)| + h(\tau)|z'(\tau)|)\,d\tau.$$

Using (2.9) this becomes

$$|z(t)| \leqslant C|z(0)| + \int_0^t G(\tau)|z(\tau)|\,d\tau, \tag{2.10}$$

where

$$G(t) = Cg(t) + {}^C/_a(|A| + g(t))h(t).$$

Obviously

$$\int_0^\infty G(t)\,dt < \infty.$$

Applying the lemma to (2.10) we get

$$|z(t)| \leqslant C|z(0)|e^{\int_0^t G(\tau)d\tau} < C|z(0)|e^{\int_0^\infty G(\tau)d\tau}. \tag{2.11}$$

Thus if $|z(0)|$ is chosen small enough, $|z(t)|$, where $t \geqslant 0$, is small, and by (2.9) so is $|z'(t)|$. Thus (2.11) and (2.9) establish Theorem IV.

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