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ON WEAK CONVERGENCE OF MEASURES AND σ-COMPLETE BOOLEAN ALGEBRAS

BY

Z. SEMADENI (POZNAŃ)

Let X be a compact (Hausdorff) space, let $C(X)$ be the space of real-valued continuous functions on X (with the supremum norm) and let $\mathfrak{M}(X)$ be the space of Radon (= finite regular Borel signed) measures on X (with the variation norm). Each $\mu \in \mathfrak{M}(X)$ determines a linear functional on $C(X)$ and $\langle \mu, f \rangle$ will denote the value of μ at f, i. e., $\langle \mu, f \rangle = \int_X f(x)\mu(dx)$. Given a sequence $\mu_n \in \mathfrak{M}(X)$, we shall say that (μ_n) converges *weakly* to μ if $\mathfrak{z}(\mu_n) \to \mathfrak{z}(\mu)$ for every linear functional $\mathfrak{z}$ over $\mathfrak{M}(X)$, and we shall say that (μ_n) *converges *weakly* to μ if $\langle \mu_n, f \rangle \to \langle \mu, f \rangle$ for every $f \in C(X)$. Weak convergence always implies *weak convergence; A. Grothendieck ([6], p. 168) has proved that *if X is compact and extremally disconnected, then both convergences are equivalent* (thus, the weak and *weak topologies are sequentially equivalent, which does not mean that they are topologically equivalent). This theorem is also valid if X is a closed subset of an extremally disconnected space (cf. [7] p. 46); if $C(X)$ admits a projection from any Banach space containing it as a subspace, then X has also this property (cf. [2], p. 109-110), but the converse statement is not true (cf. [8]).

The purpose of this note is to discuss this result and related questions. The crutial point in Grothendieck's proof is to reduce the problem to a lemma of Phillips [9]; the proof below follows this line ([1]).

Suppose that X is 0-dimensional (= totally disconnected, X being compact); then (μ_n) tends *weakly to 0 if and only if the following conditions are satisfied:

(1) $$\sup\|\mu_n\| < \infty,$$

(2) $$\mu_n(G) \to 0 \textit{ for every open-closed set } G \subset X.$$

Weak convergence requires an additional condition (cf. [3], [6], p. 146, [4], p. 306): *μ_n are equicontinuous with respect to some non-negative*

([1]) Another proof of Grothendieck's theorem was given by Ando [1].

Radon measure λ *on* X (e. g. with respect to $\lambda = \sum 2^{-n}|\mu_n|$, where $|\mu| = \mu_+ + \mu_-$).

Equicontinuity of (μ_n) means that the relation

$$\lim_{\lambda(E)\to 0} \mu_n(E) = 0, \tag{3}$$

holds uniformly with respect to n; if S is the quotient space of Borel subsets of X modulo λ-null sets, provided with the metric $\varrho(A, B) = \lambda(A\setminus B)+\lambda(B\setminus A)$, then this condition means that μ_n (considered as real-valued functions on S) are equicontinuous with respect to this metric, i. e.,

$$\lim_{\delta\to 0+} \sup_n \omega_{\mu_n}(\delta) = 0,$$

where ω_μ denotes the modulus of continuity of μ defined as usually by $\omega_\mu(\delta) = \sup\{|\mu(A)-\mu(B)| : \varrho(A, B) \leqslant \delta\}$. If T is a dense subset of S and $\overline{\omega}_\mu$ denotes the modulus of continuity of μ restricted to T, then $\omega_\mu(\delta) \geqslant \overline{\omega}_\mu(\delta) \geqslant \omega_\mu(\delta')$ for any $\delta > \delta' > 0$. Consequently, we may assume in (3) that E is open-closed (X being 0-dimensional). At the same time, if (μ_n) tends weakly to 0, then μ_n are equicontinuous on S and converge on a dense subset (viz. for all open-closed subsets of X), whence

(4) $\quad \mu_n(E) \to 0$ *for every Borel set in* X.

Conversely, by Vitali-Hahn-Saks theorem, (4) implies the equicontinuity of μ_n with respect to $\lambda = \sum 2^{-n}|\mu_n|$, and conditions (1) and (4) are necessary and sufficient in order that μ_n tend weakly to 0 (cf. [4], p. 308).

Thus, Grothendieck's theorem may be roughly stated as follows: If X is "highly disconnected", then there are enough open-closed sets to make (1) and (2) imply (4). It is not known, however, what type of disconnectedness is necessary and sufficient in order that Grothendieck's theorem be valid. In this paper we give some necessary and some sufficient conditions (²).

We shall consider the following conditions:

(i) *X is a closed subset of a compact* 0*-dimensional space Y which has the following property: the closure of the union of any countable family of open-closed subsets of Y is open.*

This means (cf. [13], p. 21, 30, 73) that X is the Stone space of a quotient of a σ-complete Boolean algebra. E. g., the Stone-Čech compacti-

(²) Recently Seever [11] obtained some interesting results concerning Grothendieck's theorem. In particular, he showed that the compact connected space $\beta R^+\setminus R^+$, where $R^+ = \{t: 0 \leqslant t < \infty\}$, does have Grothendieck's property, too. (Cf. Gillman and Jerison [5], p. 211).

fication βN of the set N of positive integers and $\beta N \setminus N$ satisfy (i), though $\beta N \setminus N$ is the Stone space of the quotient algebra of all subsets of N modulo finite sets, which is not σ-complete (cf. [12] and [13], p. 53).

(ii) *$A_1, A_2, \ldots$ being any non-empty disjoint open-closed subsets of X, there exists a linear bicontinuous operator $T: C(\beta N) \to C(X)$ assigning to every bounded real sequence $c_1, c_2, \ldots$ (considered as an element of $C(\beta N)$) a function $f \in C(X)$ such that $f(x) = c_n$ for $x \in A_n$ $(n = 1, 2, \ldots)$.*

If this is the case, then T is an isomorphic embedding of $C(\beta N)$ into $C(X)$ and for every bounded sequence $c_1, c_2, \ldots$ the function f constant on each A_n and assuming the value c_n on A_n can be continuously extended to the closure of $\bigcup A_n$ and, finally, all such extended functions can be simultaneously extended from $\overline{\bigcup A_n}$ to the whole of X, which means that the extension of the sum of two functions is equal to the sum of extensions. (In the construction of T below, T is also a lattice homomorphism, i. e. $T(g \vee h) = Tg \vee Th$ for all $g, h \in C(\beta N)$, and the norms are preserved.)

(iii) *If (A_m) is any sequence of disjoint open-closed subsets of X and (μ_n) tends *weakly to 0, then*

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} |\mu_n(A_m)| = 0. \tag{5}$$

If $X = \beta N$ and A_m is the one-point set (m) for $m \in N$, then (iii) is the statement of the quoted lemma of Phillips ([9], p. 525, [2], p. 32).

(iv) *Every *weakly convergent sequence of Radon measures on X is weakly convergent* (Grothendieck's property).

(v) *Given any sequence $x_1, x_2, \ldots$ of distinct points of X and an infinite matrix (a_{ik}) defining a regular (i. e. limit-preserving) method of summability, there exists a function $f \in C(X)$ such that the sequence $f(x_1), f(x_2), \ldots$ is not limitable by the method, i. e., the sequence of numbers*

$$b_n = \sum_{k=1}^{\infty} a_{nk} f(x_k)$$

is divergent.

W. Rudin ([10], p. 200) has proved that βN has the property (v) in the case when (a_{nk}) is the Cesàro matrix, i. e., when b_n is the arithmetic mean of the values of f at $x_1, \ldots, x_n$.

THEOREM. *If X is compact and 0-dimensional, then the implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) *hold.*

Proof. Assume (i). Let A_n $(n = 1, 2, \ldots)$ be disjoint sets open-closed in X. There exist disjoint sets B_1 and H_1 open-closed in Y and such that

$A_1 = B_1 \cap X$ and $A_n \subset H_1$ for $n \geqslant 2$. Repeating this argument, we find disjoint sets $B_1, B_2, \ldots$ open-closed in Y and such that $A_n = B_n \cap X$ for $n = 1, 2, \ldots$ The sets

$$h(M) = \overline{\bigcup_{m \in M} B_m}$$

are open-closed in Y for all subsets M of N. Since B_n's are disjoint, h is a one-one Boolean homomorphism from the algebra $\mathfrak{N}$ of subsets of N into the algebra $\mathfrak{A}$ of open-closed subsets of $Y_0 = h(N)$; it induces a continuous map φ_0 from Y_0 onto βN, which can be extended to a continuous map $\varphi_1 \colon Y \to \beta N$, because Y_0 is open-closed in Y. Since $\varphi_0^{-1}(m) = B_m$ for any $m \in N$, φ_1 maps X onto βN; let φ be the restriction of φ_1 to X. This map φ induces, in turn, the desired operator T, defined by $Tg(x) = g(\varphi x)$ for $x \in X$ and $g \in C(\beta N)$, which satisfies (ii).

(ii) implies (iii) by Phillip's lemma (cf. [2], p. 33, Cor. 1).

(iii) implies (iv) by a criterion of weak convergence due to Grothendieck ([6], p. 147, condition (3) applied to open-closed subsets), because (5) implies $\mu_n(A_n) \to 0$ as $n \to \infty$ (cf. also [6], p. 168).

Finally, assume (iv). Let $E = \{x_1, x_2, \ldots\}$ be any countable set of distinct points of X, and let

$$\mu_n = \sum_{k=1}^{\infty} a_{nk} \delta_{x_k}, \qquad \|\mu_n\| = \sum_{k=1}^{\infty} |a_{nk}|,$$

where δ_x denotes the Dirac measure concentrated at x; since (a_{ik}) is a regular method, μ_n are finite measures and (1) is satisfied (cf. [4], p. 75). If $\langle \mu_n, f \rangle$ converged for every $f \in C(X)$, (μ_n) would converge weakly, whence, by (4), $\mu_n(T)$ would converge for every subset T of E. In other words, every zero-one sequence would be summable by the method (a_{ik}), whence every bounded sequence would be summable, contradicting a well known theorem of Schur, because the sequences $(a_{ik})_{k=1,2,\ldots}$, considered as elements of the space l, would converge weakly in l, whence strongly (cf. [2], p. 33) and (a_{ik}) could not be regular.

Added in proof. Property (v) is related to those investigated in a recent paper by Henriksen and Isbell [14].

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