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DIMENSION AND MAPPINGS OF SPACES WITH FINITE DEFICIENCY

BY

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W. Hurewicz proved in 1927 the well known theorem on mappings which lower dimension. Originally, this theorem concerned mappings of compacta. In a generalized form (see [2], p. 91) it states that if a mapping of a space is closed, then the dimension of the space does not exceed the sum of the dimension of its image and the dimension of the mapping. Evidently, the closedness of the mapping is a necessary hypothesis here, but weaker inequalities may be proved without it. In the present note we shall show two theorems in this direction (see 3.1 and 3.2); instead of the closedness of the mapping some other conditions will be assumed. One of them is the local compactness of the mapping, and there are also restrictions on the deficiency and what I call an "inductive invariant" of the space.

J. de Groot was the first to consider an invariant of this kind in connection with the notion of deficiency, introduced by him in 1942. For a separable metric space X, the *deficiency* $\operatorname{def} X$ is defined (see [1], p. 50) to be the least dimension of a set which can compactify X, i. e.

$$\operatorname{def} X = \min\{\dim(\overline{X}-X) : \overline{X} \in \boldsymbol{C}(X)\},$$

where $\boldsymbol{C}(X)$ denotes the family of all metrizable compactifications of X. The inductive invariants appear in a natural way as a generalization of the inductive dimension. They can be specialized for concrete purposes, e. g. the invariant "$\operatorname{com} X$" in our terminology arises as a hypothetical intrinsic characterization of the deficiency of a space (see [1], p. 51).

S. Mazurkiewicz constructed in 1927 an n-dimensional subset M of the Cartesian product $T \times S^n$ of the Cantor ternary set T and the n-dimensional sphere S^n ($n = 1, 2, \ldots$) such that M is a G_δ-set and the projection of $T \times S^n$ onto T, restricted to M, is an 1-1 mapping (see [5],

p. 319). Suppose A is an n-dimensional subset of a compactum X and C is a component of X. Then there are two "traces" of A on C: the set $A \cap C$ and the set $C \cap \overline{A-C}$. We shall prove (see 4.1) that for some component of X at least one of such traces is at least n-dimensional. This follows from a formula concerning closed mappings.

Yu. M. Smirnov noticed in a letter to me in 1962 that, given a closed mapping f of a space X and a subset A of X, there exists a minimal set $A^f \subset X$ such that $A \subset A^f$ and f restricted to A^f is closed (see 1.1). Hence, though the set function $F(A) = A^f$ is clearly not additive, and consequently it does not satisfy the axioms of closure, the set A^f could be called the "closure of A relative to f". We shall give (see 1.2) a formula for A^f.

T. Nishiura has recently generalized some my results concerning the dimension of quasi-components (see [7], p. 9). If follows, for instance, that

$$\dim X \leqslant \operatorname{def} X$$

for every non-compact space X whose quasi-components are locally compact and 0-dimensional. Observe that the above inequality gives a sharpening of a result obtained by S. Mazurkiewicz in 1934 for spaces whose quasi-components were single points (see [6], p. 267). The theorems of T. Nishiura can, however, be sharpened and generalized further on (see 3.2 and 4.2).

All spaces considered throughout will be separable metric. The closure of a set A will be denoted by $\overline{A}$.

1. Closed mappings. A mapping f of a space X is said to be *closed* if for every closed set C in X the set $f(C)$ is closed in $f(X)$.

Suppose f is a closed mapping of X and $A \subset X$ is an arbitrary set. We shall denote by A^f the common part of all subsets Z of X such that $A \subset Z$ and $f|Z$ is a closed mapping.

1.1. *If f is a closed mapping of a space X, then the mapping $f|A^f$ is closed for every $A \subset X$. Hence A^f is a minimal set in which A is contained and on which f is closed.*

Proof. Take a closed subset C of A^f and a point y belonging to $f(A^f) \cap \overline{f(C)}$. We have to prove that $y \in f(C)$. So we can assume that there exist points $y_i \in f(C)$ converging to y and different from y, whence $y_i = f(x_i)$ and $x_i \in C$ for $i = 1, 2, \ldots$ The image $f(Q)$ of the set $Q = \{x_i: i = 1, 2, \ldots\}$ is not closed in $f(X)$, and consequently the set Q is not closed in X, i.e. there exists a subsequence x_{i_j} converging to a point $x \in X$.

If $Z \subset X$ is a set such that $A \subset Z$ and $f|Z$ is a closed mapping, then

$$x_{i_j} \in C \subset A^f \subset Z \quad \text{and} \quad y_{i_j} \neq y \in f(A^f) \subset f(Z)$$

for $j = 1, 2, \ldots$ Hence the set $f(Q')$, where $Q' = \{x_{i_j}: j = 1, 2, \ldots\}$, is not closed in $f(Z)$, and consequently Q' is not a closed subset of Z; therefore $x \in Z$.

We get $x \in A^f$. Since $x \in \overline{C}$ and C is a closed subset of A^f, we have $x \in C$; but clearly $y = f(x)$, which yields $y \in f(C)$.

1.2. *If f is a closed mapping of a space X, then for every $A \subset X$ the set A^f is given by the formula*

$$A^f = A \cup \bigcup_{y \in f(A)} [f^{-1}(y) \cap \overline{A - f^{-1}(y)}].$$

Proof. Let us denote by P the set on the right side of the formula. Evidently, $f(P) = f(A)$. We first prove that $f|P$ is a closed mapping. Indeed, suppose on the contrary that there exists a set $C \subset P$ closed in P, and a point

$$y_0 \in [\overline{f(C)} \cap f(P)] - f(C).$$

Then $C \subset P - f^{-1}(y)$ and $y_0 \in \overline{f(C)} \subset \overline{f(\overline{C})} = f(\overline{C})$, the mapping f being closed. It follows that $y_0 = f(x_0)$, where $x_0 \in \overline{C}$. Since

$$P \subset \overline{A} \subset \overline{A - f^{-1}(y_0)} \cup f^{-1}(y_0),$$

we obtain $C \subset \overline{A - f^{-1}(y_0)}$, and consequently

$$x_0 \in f^{-1}(y_0) \cap \overline{C} \subset f^{-1}(y_0) \cap \overline{A - f^{-1}(y_0)},$$

which, together with the condition $y_0 \in f(P) = f(A)$, implies $x_0 \in P$. But the set C being closed in P, we have $x_0 \in C$, whence $y_0 \in f(C)$, and this is a contradiction. The mapping $f|P$ is thus closed, and so $A^f \subset P$.

Now, let Z be a subset of X such that $A \subset Z$ and $f|Z$ is a closed mapping. If $x \in f^{-1}(y) \cap \overline{A - f^{-1}(y)}$, where $y \in f(A)$, then there are points $x_i \in A - f^{-1}(y)$ converging to x $(i = 1, 2, \ldots)$. Put $Q = \{x_i: i = 1, 2, \ldots\}$. Since $Q \subset Z$ and

$$y = f(x) \in [\overline{f(Q)} \cap f(Z)] - f(Q),$$

the set Q is not closed in Z, which gives $x \in Z$. We infer that $P \subset Z$, and therefore $P \subset A^f$.

Remark. Theorems 1.1 and 1.2 are not true for non-metrizable spaces. In fact, consider the Čech-Stone compactification βN of the discrete countable infinite space N, an arbitrary point $a \in \beta N - N$ and the natural mapping f of βN onto the one-point compactification of N. If $A = N \cup \{a\}$, we have $A^f = A$ and the mapping $f|A$ is not closed. If $A = \beta N - \{a\}$, we have $A^f = A$, the mapping $f|A^f$ is closed, but the set on the right side in the formula from 1.2 is equal to βN.

2. Inductive invariants. Let $\boldsymbol{F}$ be a topologically closed family of spaces, i. e. such that for every $F \in \boldsymbol{F}$ the family $\boldsymbol{F}$ contains all spaces homeomorphic to F. The *inductive invariant* $I(X, \boldsymbol{F})$ induced by $\boldsymbol{F}$ is defined for every space X as follows: $I(X, \boldsymbol{F}) = -1$ if and only if $X \in \boldsymbol{F}$, and $I(X, \boldsymbol{F}) \leqslant n$ provided that each point of X has arbitrarily small open neighbourhoods U in X such that $I(\overline{U} - U, \boldsymbol{F}) \leqslant n-1$. For instance, if $\boldsymbol{C}_0$ is the family consisting only of an empty space 0, we have $\dim X = I(X, \boldsymbol{C}_0)$. Observe that if $\boldsymbol{F} \subset \boldsymbol{F}'$, then $I(X, \boldsymbol{F}') \leqslant I(X, \boldsymbol{F})$ for every X; this readily follows by induction on $I(X, \boldsymbol{F})$.

Consider the families $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ consisting of all compact spaces and of all locally compact ones, respectively. Further, let $\boldsymbol{C}_3$ be the family of all spaces X such that X contains a compact subset C with $\dim C = \dim X$. To denote the inductive invariants $I(X, \boldsymbol{C}_i)$, induced by $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, and $\boldsymbol{C}_3$, we shall use the symbols $\operatorname{com} X$, $\operatorname{loccom} X$, and $\operatorname{subcom} X$, respectively. Since $\boldsymbol{C}_0 \subset \boldsymbol{C}_1 \subset \boldsymbol{C}_2$, we have

$$\operatorname{loccom} X \leqslant \operatorname{com} X \leqslant \dim X$$

for every X. Each finite-dimensional space from $\boldsymbol{C}_2$ belongs to $\boldsymbol{C}_3$, which implies that

$$\operatorname{subcom} X \leqslant \operatorname{loccom} X$$

for every X with $\dim X < \infty$. Since $\boldsymbol{C}_1 \subset \boldsymbol{C}_3$, it follows that

$$\operatorname{subcom} X \leqslant \operatorname{com} X$$

for every X. The inequality $\operatorname{com} X \leqslant 0$ means that the space X is peripherically compact, whence

$$\operatorname{com} X \leqslant \operatorname{loccom} X + 1$$

for every X. There are also the inequalities

$$\operatorname{com} X \leqslant \operatorname{def} X \leqslant \dim X,$$

the first of whose can easily be verified by induction on $\operatorname{def} X$, and the second is a consequence of the Hurewicz theorem (see [2], p. 65). As far as I know, the problem whether $\operatorname{com} X = \operatorname{def} X$, raised by J. de Groot 21 years ago (see [1], p. 51), remains unsolved.

Finally, denoting by 2_c^X the collection of all compact subsets of the space X, we get

$$\dim X \leqslant \operatorname{subcom} X + \sup\{\dim C \colon C \in 2_c^X\} + 1$$

for every X. The last inequality is verifiable without difficulty, by induction on $\operatorname{subcom} X$.

3. Dimension of mappings. Suppose f is a mapping of a space X. For a numerical invariant $N(X)$ of spaces, e. g. $N(X) = \dim X$, or $N(X) = \operatorname{com} X$, we shall denote by $N(f)$ the number

$$N(f) = \sup\{N(f^{-1}(y)) \colon y \in f(X)\}.$$

If the invariant $N(X)$ is monotone with respect to closed subsets of X, which holds for all the invariants described in the preceding section, except the invariant $\operatorname{subcom} X$, then $N(f) \leqslant N(X)$.

3.1. *If f is a mapping of a space X, then*

$$\dim X \leqslant \dim f(X) + \dim f + \operatorname{subcom} X + 1.$$

Proof. Let C be an arbitrary compact subset of X. Since $f|C$ is a closed mapping, it follows from the Hurewicz theorem (see [2], p. 91) that

$$\dim C \leqslant \dim f(C) + \dim f|C \leqslant \dim f(X) + \dim f,$$

which yields 3.1 by virtue of the last inequality from the preceding section.

The mapping f will be called *locally compact* if $\operatorname{loccom} f = -1$, i. e. if the set $f^{-1}(y)$ is locally compact for every $y \epsilon f(X)$.

3.2. *If f is a locally compact mapping of a space X, then*

$$\dim X \leqslant \dim f(X) + \max\{\dim f, \operatorname{def} X\}.$$

Proof. Let $\bar{X}$ be a compactification of X such that $\dim(\bar{X} - X) = \operatorname{def} X$. The projection g of the product $\bar{X} \times f(X)$ onto the space $f(X)$ is a closed mapping (see [3], p. 4). The graph

$$G = \{(x, f(x)) : x \epsilon X\} \subset \bar{X} \times f(X)$$

of the mapping f being closed in $X \times f(X)$, we have $\bar{G} - G \subset (\bar{X} - X) \times f(X)$, whence

$$\begin{aligned}(g|\bar{G})^{-1}(y) &= \bar{G} \cap g^{-1}(y) = \bar{G} \cap (\bar{X} \times \{y\}) \\ &= (f^{-1}(y) \times \{y\}) \cup ((\bar{G} - G) \cap (\bar{X} \times \{y\})) \subset (f^{-1}(y) \cup (\bar{X} - X)) \times \{y\}\end{aligned}$$

for every $y \epsilon f(X)$. Put $F_y = f^{-1}(y)$. Since the set F_y is locally compact and closed in X, it is open in its closure $\bar{F}_y$ in $\bar{X}$ and $\bar{F}_y - F_y \subset \bar{X} - X$. It follows that

$$\dim \bar{F}_y = \max\{\dim F_y, \dim(\bar{F}_y - F_y)\} \leqslant \max\{\dim f, \operatorname{def} X\},$$

which implies

$$\dim(\overline{f^{-1}(y)} \cup (\bar{X} - X)) = \max\{\dim \bar{F}_y, \operatorname{def} X\} \leqslant \max\{\dim f, \operatorname{def} X\}$$

for every $y \epsilon f(X)$. Thus we get

$$\dim (g|\bar{G})^{-1}(y) \leqslant \max\{\dim f, \operatorname{def} X\}$$

for every $y \epsilon f(X)$, i. e. $\dim g|\bar{G} \leqslant \max\{\dim f, \operatorname{def} X\}$. But $g|\bar{G}$ is a closed mapping and the graph G is homeomorphic to X. We infer by the same Hurewicz theorem that

$$\dim X \leqslant \dim \bar{G} \leqslant \dim g(\bar{G}) + \dim g|\bar{G} \leqslant \dim f(X) + \max\{\dim f, \operatorname{def} X\}.$$

Remark. Theorem 3.2 generalizes Theorem 1 from [4], as well as Theorem 2 from [7], where as f some special mapping of the space X is taken; namely, the sets $f^{-1}(y)$ coincide with the quasi-components of X. The estimations of the dimension, given in Theorems 3.1 and 3.2, use internal and external approximations of the space X by compacta, respectively. Really, those compacta range over the collection 2_c^X in 3.1, and the collection $C(X)$ in 3.2. However, there is in Theorem 3.2 an additional hypothesis that the mapping is locally compact. A more full analogy between internal and external cases can probably be done by introducing a suitable inductive invariant in 3.2. In this connection a question remains unsolved (**P 469**) whether the inequality

$$\dim X \leqslant \dim f(X) + \max\{\dim f, \operatorname{def} X\} + \operatorname{loc}\operatorname{com} f + 1$$

holds for every mapping f of the space X.

4. Dimension of components. The following theorem refers to the Mazurkiewicz example mentioned at the beginning of the paper:

4.1. *If A is a subset of a compact space X such that* $\dim A \cap C < \dim A$ *for every component C of X, then there exists a component C_0 of X such that*

$$\dim A \leqslant \dim C_0 \cap \overline{A - C_0}.$$

Proof. Let f be the natural mapping of X onto the space of components of X. Since $f|A^f$ is a closed mapping according to 1.1, we have

$$\dim A \leqslant \dim f|A^f,$$

by the Hurewicz theorem and the equality $\dim f(X) = 0$. Consequently, 1.2 implies that

$$\dim A \leqslant \dim\big((A \cap C_0) \cup (C_0 \cap \overline{A - C_0})\big)$$

for some $C_0 = f^{-1}(y_0)$. The set $C_0 \cap \overline{A - C_0}$ being compact and $\dim A \cap C_0 < \dim A$, we get the required inequality.

4.2. *If every component C of a space X has a dimension* $\dim C \leqslant n$, *then*

$$\dim X \leqslant n + \operatorname{sub}\operatorname{com} X + 1.$$

Proof. The statement is obvious if $\operatorname{sub}\operatorname{com} X = -1$. Suppose $\operatorname{sub}\operatorname{com} X$ is finite and 4.2 holds for spaces Y with $\operatorname{sub}\operatorname{com} Y < \operatorname{sub}\operatorname{com} X$. Then each point of X has arbitrarily small open neighbourhoods U in X such that

$$\operatorname{sub}\operatorname{com}(\overline{U} - U) \leqslant \operatorname{sub}\operatorname{com} X - 1$$

and the dimension of components of the set $\overline{U} - U$ does not exceed n; thus

$$\dim(\overline{U} - U) \leqslant n + \operatorname{sub}\operatorname{com}(\overline{U} - U) + 1 \leqslant n + \operatorname{sub}\operatorname{com} X.$$

Remark. Since $\operatorname{subcom} X \leqslant \operatorname{com} X \leqslant 0$ for peripherically compact spaces, Theorem 4.2 is a generalization of Theorem 2 from my paper [4]. On the other hand, each component is contained in a quasi-component, and therefore the hypothesis in 4.2 can be replaced by the condition that $\dim Q \leqslant n$ for every quasi-component Q of X. Theorem 4.2 with this stronger hypothesis is a sharpening of Theorem 3 from paper [7] of T. Nishiura, where the dimension of X is estimated from above by the sum $n+\operatorname{def} X+1$.

REFERENCES

[1] J. de Groot, *Topologische Studiën*, Assen 1942.

[2] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton 1948.

[3] C. Kuratowski, *Topologie II*, Warszawa 1961.

[4] A. Lelek, *On the dimension of quasi-components in peripherically compact spaces*, Colloquium Mathematicum 9 (1962), p. 241-244.

[5] S. Mazurkiewicz, *Sur les problèmes $\varkappa$ et λ de Urysohn*, Fundamenta Mathematicae 10 (1927), p. 311-319.

[6] — *Über total zusammenhangslose Mengen*, Fundamenta Mathematicae 22 (1934), p. 267-269.

[7] T. Nishiura, *On the dimension of semi-compact spaces and their quasi-components*, Colloquium Mathematicum 12 (1964), p. 7-10.

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