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OPEN AND IMAGE-OPEN RELATIONS

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Although one frequently encounters, in the literature of continuous relations (see [3] for a bibliography), the hypothesis that a relation be a closed set or that the image of each point be closed or compact, the hypothesis of openness of a relation or of the image of each point (discussed briefly by Choquet [1]) occurs very rarely. This absence is not at all surprising in light of the results of this note, i. e., such relations, when upper-semicontinuous, are "almost constant".

If $T \subseteq X \times Y$ is any relation and $x \in X$, let $T(x) = \{y \mid y \in Y$ and $(x, y) \in T\}$ and $D(T) = \{x \mid x \in X$ and $T(x) \neq \emptyset\}$. $T(x)$ is called the *image* of x under T and $D(T)$ the *domain* of T. Let $R(X, Y) = \{T \mid T \subseteq X \times Y$ and $D(T) = X\}$.

If X and Y are topological spaces and $T \in R(X, Y)$, T is *upper-semicontinuous* at $x_0 \in X$ iff for every neighborhood V of $T(x_0)$, there is a neighborhood U of x_0 such that $x \in U$ implies that $T(x) \subseteq V$. T is *lower-semicontinuous* at x_0 iff for each $y \in T(x_0)$ and for each neighborhood V of y, there is a neighborhood U of x_0 such that $x \in U$ implies $T(x) \cap V \neq \emptyset$. T is *open* iff T is an open subset of $X \times Y$ and is *image-open* iff $T(x)$ is open in Y for each $x \in X$.

PROPOSITION 1. *If $T \in R(X, Y)$ is open and upper-semicontinuous on X, then T is constant on each component of X.*

Proof. Assume that X is connected. For any $x_0 \in X$, $T(x_0)$ is a neighborhood of itself and hence there is a neighborhood U of x_0 such that $\bigcup\{T(x) \mid x \in U\} = T(x_0)$. Let U_0 be the union of all such neighborhoods of x_0. For each $y \in T(x_0)$, let $F(y) = \{x \mid x \in U_0$ and $y \notin T(x)\}$. Since T is open, there exists an open neighborhood W of x_0 contained in U_0 such that $W \cap F(y) = \emptyset$. Let $W(y)$ be the maximal such neighborhood. But $W(y)$ is open and closed and hence by connectedness $W(y) = X$. Hence $x \in X$ implies $T(x) = T(x_0)$. The proposition follows by applying this result to each component of an arbitrary X.

For each $T \in R(X, Y)$ define $T' \in R(X, Y)$ by $T'(x) = \overline{T(x)}$ for all $x \in X$. (In general, it is not the case that T' is the closure of T in $X \times Y$ [2].)

PROPOSITION 2. *If $T \in R(X, Y)$ is upper- and lower-semicontinuous and image-open, then T' is constant on each component of X.*

Proof. Assume X to be connected. Choose any $x_0 \in X$ and let $G = \{x \mid x \in X \text{ and } T(x) \subseteq \overline{T(x_0)}\}$. Since T is upper-semicontinuous and image open, G is open, and, since T is also lower-semicontinuous, G is closed. Hence, by connectedness, $G = X$. Since x_0 was arbitrary, T' is constant on X. Applying this result to an arbitrary space yields the proposition.

Since every open relation is both lower-semicontinuous and image-open, both the hypothesis and the conclusion of Proposition 2 are somewhat weaker than those of Proposition 1. It is also possible to omit the hypothesis of lower-semicontinuity and substitute restrictions on the space X as will be done in the next result.

If $T \in R(X, Y)$, a non-empty open subset A of X is a *neighborhood of constancy* of T iff $T(x) = T(x')$ for all $x, x' \in A$.

PROPOSITION 3. *Let X be locally countably compact and regular (T_3). If $T \in R(X, Y)$ is image-open and upper-semicontinuous, then $X = E \cup F$, where E is a union of neighborhoods of constancy of T and F is nowhere dense in X.*

Proof. For any $x_0 \in X$, let U_0 be a neighborhood of x_0 such that $T(U_0) = \bigcup \{T(x) \mid x \in U_0\} \subseteq T(x_0)$. Each such U_0 intersects a neighborhood of constancy of T. For if not, take K a countably compact neighborhood of x_0 and let $V = K \cap U_0$. Then there exists an $x_1 \in V$ such that $T(x_1)$ is a proper subset of $T(x_0)$, and a closed neighborhood U_1 of x_1 such that $U_1 \subseteq V$ and $T(U_1) \subseteq T(x_1)$. Since $U_1 \subseteq U_0$, it can intersect no neighborhood of constancy. Hence the argument may be repeated countably many times, generating sequences $\{x_n\}$ and $\{U_n\}$ such that for each $n > 0$, $T(x_n)$ is a proper subset of $T(x_{n-1})$, U_n is closed, $U_n \subseteq U_{n-1} \cap K$, and $T(U_n) \subseteq T(U_{n-1})$. Since the x_n are distinct, there is an accumulation point y of $\{x_n\}$ belonging to $\bigcap_n U_n$. Then, for all n, $T(y)$ is a proper subset of $T(x_n)$. Hence there is no neighborhood W of y such that $T(W) \subseteq T(y)$, contradicting the upper-semicontinuity of T.

Since neighborhoods of the type of U_0 form a basis, the union E of all neighborhoods of constancy of T is dense in X and thus $F = X \setminus E$ is nowhere dense.

REFERENCES

[1] G. Choquet, *Convergences*, Annales de l'Université de Grenoble, Section des Sciences Math. et Phys., **23** (1947-8), p. 58-112.

[2] S. P. Franklin and R. H. Sorgenfrey, *Closed and image-closed relations*, to appear.

[3] W. L. Strother, *Continuous multi-valued functions*, Boletin de la Sociedad Matemática de Sao Paulo 10 (1955), p. 87-120.

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