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COMMUTATIVE REGULAR SEMIGROUPS

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Dubikajtis and Jarek [3] have investigated connection between commutative regular semigroups and elementary pseudogroups ([1]).

This paper is devoted to the study of the structure of commutative regular semigroups and their homomorphisms. I show that it is possible to associate with every commutative regular semigroup a direct system of Abelian groups over a semilattice. This correspondence enables us to describe all homomorphic images of a commutative regular semigroup and thus to generalize the well known theorem concerning homomorphic images of Abelian groups.

§ 1. Partially ordered sets. Direct systems of Abelian groups. A set T is said to be *partially ordered* if there is in T a binary relation $\leqslant$ which is defined for certain (not necessarily for all) pairs $t_1, t_2 \in T$ and which is reflexive, transitive and antisymmetric.

A partially ordered set T is said to be a *directed set* if for any $t_1, t_2 \in T$ there exists an element $t_3 \in T$ such that $t_1 \leqslant t_3$ and $t_2 \leqslant t_3$.

A partially ordered set T is called a *semilattice* if for any $t_1, t_2 \in T$ there exists their smallest upper bound $t_1 + t_2 \in T$ ([2]).

Any semilattice is of course a directed set.

Let T be a partially ordered set and $\sim$ an equivalence relation in T (i. e. a binary relation, which is reflexive, transitive and symmetric). We shall denote by $\tilde{t}$ the equivalence class of t, i. e., the set of all elements $t' \in T$ such that $t' \sim t$, and by $\tilde{T}$ the set of all equivalence classes $\tilde{t}$.

It is well known (see [4]) that if the relation $\sim$ satisfies the conditions

([1]) The notion of an elementary pseudogroup has been introduced by Dubikajtis in [2].

([2]) We shall use the notation $t_1 + t_2$ for the smallest upper bound of t_1 and t_2. This shall be convenient, since all semilattices considered in this paper are semigroups with addition as the group operation, and the smallest upper bound always coincides with the sum of two elements.

(a) if $t_1 \leqslant t \leqslant t_2$ and $t_1 \sim t_2$, then $t \sim t_2$,

(b) if $t_1 \leqslant t_2$, then for every $t_1' \sim t_1$ there exists an element $t_2' \sim t_2$ such that $t_1' \leqslant t_2'$,

then the ordering relation $\leqslant$ induces a partially ordering relation in $\tilde{T}$, defined as follows:

$\tilde{t}_1 \leqslant \tilde{t}_2$ if and only if there exist elements $t_1' \sim t_1$ and $t_2' \sim t_2$ such that $t_1' \leqslant t_2'$.

An equivalence relation $\sim$ defined in a semilattice T is said to be a *congruence relation*, if

(c) for any $t_1, t_1', t \in T$ if $t_1 \sim t_1'$, then $t_1 + t \sim t_1' + t$.

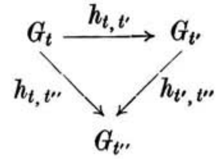
It is evident that a congruence relation satisfies conditions (a) and (b) and that the set $\tilde{T}$ is again a semilattice. Moreover, the smallest upper bound of $\tilde{t}_1$ and $\tilde{t}_2$ in $\tilde{T}$ is $\widetilde{t_1 + t_2}$ (see [1], p. 44).

Two partially ordered sets T and S are called *similar* if there exists a one-to-one transformation φ of T onto S such that for any $t_1, t_2 \in T$ the relation $t_1 \leqslant t_2$ holds if and only if $\varphi(t_1) \leqslant \varphi(t_2)$. φ will be called a *similarity-function* of T and S.

Let T be a directed set. Suppose that to every $t \in T$ there corresponds an Abelian group G_t, and to every pair $t, t' \in T$ such that $t \leqslant t'$ there corresponds a homomorphism $h_{t,t'}$ of G_t in $G_{t'}$. The system $\{G_t, h_{t,t'}\}_{t \in T}$ of groups G_t and homomorphisms $h_{t,t'}$ is said to be a *direct system* over the directed set T, if

(1) for any $t \in T$, $h_{t,t}$ is the identity isomorphism of G_t,

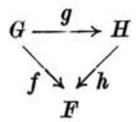
(2) for any $t, t', t'' \in T$ such that $t \leqslant t' \leqslant t''$ the diagram

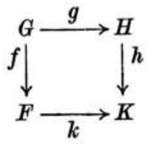


is commutative ([3]).

Two direct systems $\{G_t, h_{t,t'}\}_{t \in T}$ and $\{F_s, f_{s,s'}\}_{s \in S}$ over directed sets T and S are called *isomorphic*, if there exists a similarity-function $\varphi_0 \colon T \to S$ and a family of mappings $\{\varphi_t\}_{t \in T}$ such that

([3]) Commutativity of diagrams





means that $f = hg$ and $kf = hg$ respectively.

1° φ_t is for any $t \in T$ an isomorphism of G_t on $F_{\varphi_0(t)}$,

2° for any $t \leqslant t'$ the diagram

$$\begin{array}{ccc} G_t & \xrightarrow{h_{t,t'}} & G_{t'} \\ \varphi_t \downarrow & & \downarrow \varphi_{t'} \\ F_s & \xrightarrow[f_{s,s'}]{} & F_{s'} \end{array}$$

where $\varphi_0(t) = s$, $\varphi_0(t') = s'$, is commutative.

A direct system $\{G_t, h_{t,t'}\}_{t\in T}$ over T will be said to be a *regular system*, if T is a semilattice and the groups G_t are mutually disjoint.

Suppose that to every $t \in T$ there corresponds a subgroup H_t of G_t such that

(3) for any $t, t' \in T$ if $t \leqslant t'$, then $h_{t,t'}(H_t) \subset H_{t'}$.

Then, the subgroups H_t together with the homomorphisms $h_{t,t'}$ restricted to H_t form again a direct system over T, called the *subsystem* of the system $\{G_t, h_{t,t'}\}_{t\in T}$ ([4]).

Let $\{H_t, h_{t,t'}\}_{t\in T}$ be a subsystem of the direct system $\{G_t, h_{t,t'}\}_{t\in T}$. In view of (3), every homomorphism $h_{t,t'}$ induces a homomorphism $\overline{h}_{t,t'}$ of the factor group G_t/H_t in $G_{t'}/H_{t'}$. The family of factor groups G_t/H_t together with the induced homomorphisms $\overline{h}_{t,t'}$ form again a direct system $\{G_t/H_t, \overline{h}_{t,t'}\}_{t\in T}$ over T, which we shall call a *factor system* of $\{G_t, h_{t,t'}\}_{t\in T}$ with respect to the subsystem $\{H_t, h_{t,t'}\}_{t\in T}$.

Let T_0 be a subset of T, which is again a directed set. We may consider the direct system composed of all these groups G_t and homomorphisms $h_{t,t'}$, whose indices belong to T_0. We shall call this system a *partial direct system* determined by the subset T_0.

Let $\{G_t, h_{t,t'}\}_{t\in T}$ be a direct system over T. $\sum_{t\in T} G_t$ will denote the direct sum of groups G_t. Let Q be the subgroup of $\sum_{t\in T} G_t$ generated by all elements of the form $g_t - h_{t,t'}(g_t)$, t and t' being arbitrary elements of T such that $t \leqslant t'$, and g_t being an arbitrary element of G_t. The factor group $G^\infty = \sum_{t\in T} G_t/Q$ will be called the *direct limit* of the direct system $\{G_t, h_{t,t'}\}_{t\in T}$.

It is well known that direct limits of isomorphic systems are isomorphic.

For any $t \in T$ a natural homomorphism h_t of G_t in G^∞ may be defined as follows: h_t is the superposition of the natural embedding of G_t in $\sum_{t\in T} G_t$ and the canonical homomorphism of $\sum_{t\in T} G_t$ on G^∞.

([4]) We shall denote the restricted homomorphisms of the subsystem by the same symbol $h_{t,t'}$.

The following properties of this homomorphism are known (see [5]):

(A) For any $t, t' \in T$ such that $t \leqslant t'$ the diagram

$$\begin{array}{ccc} G_t & \xrightarrow{h_{t,t'}} & G_{t'} \\ & {\scriptstyle h_t}\searrow \quad \swarrow{\scriptstyle h_{t'}} & \\ & G^{\infty} & \end{array}$$

is commutative.

(B) $\bigcup_{t \in T} \operatorname{Im} h_t = G^{\infty}$.

(C) $\operatorname{Ker} h_t = \bigcup_{t \leqslant t'} \operatorname{Ker} h_{t,t'}$ for any $t \in T$ (5).

(D) For any $t_1, t_2 \in T$ and $g_{t_1} \in G_{t_1}$, $g_{t_2} \in G_{t_2}$ the equality $h_{t_1}(g_{t_1}) = h_{t_2}(g_{t_2})$ holds if and only if there exists an index t such that $t_1 \leqslant t$, $t_2 \leqslant t$ and $h_{t_1,t}(g_{t_1}) = h_{t_2,t}(g_{t_2})$.

(E) Let G be an arbitrary Abelian group, $\{G_t\}_{t \in T}$ — a family of subgroups of G such that $\bigcup_{t \in T} G_t = G$ and for any groups G_t and G_s there exists a subgroup G_r containing G_t and G_s. The group G is then isomorphic with the direct limit of the direct system composed of the groups G_t and embeddings $i_{t,t'}$ of G_t in $G_{t'}$, where $G_t \subset G_{t'}$.

We shall prove the following

THEOREM 1. *Let $\{G_t, h_{t,t'}\}_{t \in T}$ and $\{F_s, f_{s,s'}\}_{s \in S}$ be two direct systems of groups over directed sets T and S respectively. Suppose that to every $t \in T$ there corresponds at least one $s \in S$ and a homomorphism $\varphi_{t,s}$ of G_t in F_s in such a way that following conditions are fulfiled:*

1° *If $\varphi_{t,s}$ is defined, then for any $s' \geqslant s$, $\varphi_{t,s'}$ is defined.*

2° *For any $t \leqslant t'$ and $s \leqslant s'$, if $\varphi_{t,s}$ and $\varphi_{t',s'}$ are defined, then the diagram*

$$\begin{array}{ccc} G_t & \xrightarrow{h_{t,t'}} & G_{t'} \\ {\scriptstyle \varphi_{t,s}}\downarrow & & \downarrow{\scriptstyle \varphi_{t',s'}} \\ F_s & \xrightarrow[f_{s,s'}]{} & F_{s'} \end{array}$$

is commutative.

Under these assumptions there exists a homomorphism $\varphi^{\infty}: G^{\infty} \to F^{\infty}$ such that the diagram

$$\begin{array}{ccc} G_t & \xrightarrow{h_t} & G^{\infty} \\ {\scriptstyle \varphi_{t,s}}\downarrow & & \downarrow{\scriptstyle \varphi^{\infty}} \\ F_s & \xrightarrow[f_s]{} & F^{\infty} \end{array} \tag{4}$$

is commutative, whenever $\varphi_{t,s}$ is defined.

(5) Im h_t and Ker h_t denote the image of G_t under the homomorphism h_t and the kernel of h_t respectively.

Proof. Let g be an arbitrary element of G^{∞}. From (B) it follows that there exists an index $t \in T$ and an element $g_t \in G_t$ such that $h_t(g_t) = g$. Let s be an arbitrary element of S such that $\varphi_{t,s}$ is defined.

Let

$$\varphi^{\infty}(g) = f_s(\varphi_{t,s}(g_t)),$$

where f_s is the natural homomorphism of F_s in F^{∞}.

We shall first prove that φ^{∞} is well defined, i. e., if for arbitrary $t_1, t_2 \in T$, $g_{t_1} \in G_{t_1}$, $g_{t_2} \in G_{t_2}$ the equality

$$h_{t_1}(g_{t_1}) = h_{t_2}(g_{t_2}) \tag{5}$$

holds, then for arbitrary $s_1, s_2 \in S$ we have

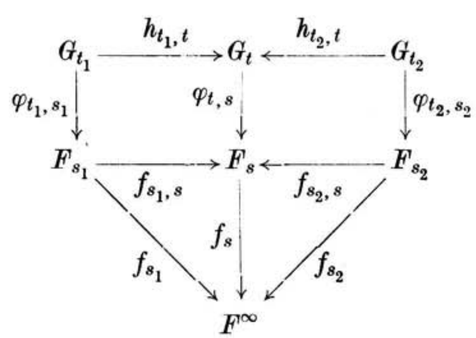
$$f_{s_1}(\varphi_{t_1,s_1}(g_{t_1})) = f_{s_2}(\varphi_{t_2,s_2}(g_{t_2})) \tag{6}$$

whenever φ_{t_1,s_1} and φ_{t_2,s_2} are defined.

Indeed, equality (5) implies, by (D), that there exists an index t such that $t_1 \leqslant t$, $t_2 \leqslant t$ and

$$h_{t_1,t}(g_{t_1}) = h_{t_2,t}(g_{t_2}). \tag{7}$$

Let s be an element of S such that $s_1 \leqslant s$, $s_2 \leqslant s$ and that $\varphi_{t,s}$ is defined (existence of such an index follows immediately from assumption 1°). Consider the diagram



in which every square and every triangle is commutative (see assumption 2° and (A)). We have $f_{s_i}(\varphi_{t_i,s_i}(g_{t_i})) = f_s[\varphi_{t_i,s}(h_{t_i,t}(g_{t_i}))]$, $i = 1, 2$, whence, by (7), equality (6) holds true.

Now we shall prove that φ^{∞} is a homomorphism. Let $g_1, g_2 \in G^{\infty}$ and let $g_{t_1} \in G_{t_1}$, $g_{t_2} \in G_{t_2}$ be two elements such that $h_{t_i}(g_{t_i}) = g_i$ $(i = 1, 2)$. Let $t \in T$ be such that $t_1 \leqslant t$ and $t_2 \leqslant t$, and let $g_t' = h_{t_1,t}(g_{t_1})$, $g_t'' = h_{t_2,t}(g_{t_2})$. From the commutativity of the diagram

$$\begin{array}{ccccc} G_{t_1} & \xrightarrow{h_{t_1,t}} & G_t & \xleftarrow{h_{t_2,t}} & G_{t_2} \\ & {\scriptstyle h_{t_1}}\searrow & \downarrow{\scriptstyle h_t} & \swarrow{\scriptstyle h_{t_2}} & \\ & & G^{\infty} & & \end{array}$$

we obtain $h_t(g_t') = h_t(h_{t_1,t}(g_{t_1})) = h_{t_1}(g_{t_1}) = g_1$ and $h_t(g_t'') = h_t(h_{t_2,t}(g_{t_2})) = h_{t_2}(g_{t_2}) = g_2$. We have $h_t(g_t'+g_t'') = g_1+g_2$, since h_t is a homomorphism. Hence, for a suitable $s \in S$, we have $\varphi^{\infty}(g_1)+\varphi^{\infty}(g_2) = f_s(\varphi_{t,s}(g_t'))+f_s(\varphi_{t,s}(g_t'')) = f_s(\varphi_{t,s}(g_t'+g_t'')) = \varphi^{\infty}(g_1+g_2)$.

To end the proof we observe that commutativity of diagram (4) is an immediate consequence of the definition of homomorphism φ^{∞}.

COROLLARY 1. *Let $\{G_t, h_{t,t'}\}_{t\in T}$, $\{F_s, f_{s,s'}\}_{s\in S}$ and $\{D_r, d_{r,r'}\}_{r\in R}$ be direct systems of groups over directed sets T, S and R respectively, $\{\varphi_{t,s}\}$, $\{\psi_{s,r}\}$ and $\{\lambda_{t,r}\}$ — families of homomorphisms of $\{G_t, h_{t,t'}\}_{t\in T}$ in $\{F_s, f_{s,s'}\}_{s\in S}$, of $\{F_s, f_{s,s'}\}_{s\in S}$ in $\{D_r, d_{r,r'}\}_{r\in R}$ and of $\{G_t, h_{t,t'}\}_{t\in T}$ in $\{D_r, d_{r,r'}\}_{r\in R}$ respectively, satisfying the assumptions of theorem 1. Moreover, suppose that for any $t\in T$, $s\in S$ and $r\in R$ the diagram*

$$(8)\qquad \begin{array}{ccc} G_t & \xrightarrow{\varphi_{t,s}} & F_s \\ & {\scriptstyle \lambda_{t,r}}\searrow \quad \swarrow{\scriptstyle \psi_{s,r}} & \\ & D_r & \end{array}$$

is commutative, whenever $\varphi_{t,s}$, $\psi_{s,r}$ and $\lambda_{t,r}$ are defined.

Then the diagram

$$\begin{array}{ccc} G^{\infty} & \xrightarrow{\varphi^{\infty}} & F^{\infty} \\ & {\scriptstyle \lambda^{\infty}}\searrow \quad \swarrow{\scriptstyle \psi^{\infty}} & \\ & D^{\infty} & \end{array}$$

where φ^{∞}, ψ^{∞} and λ^{∞} are homomorphisms defined in the proof of theorem 1, is commutative.

Proof. For an arbitrary $g\in G^{\infty}$ and $g_t\in G_t$ such that $h_t(g_t) = g$ the commutativity of diagram (8) implies $\lambda^{\infty}(g) = d_r(\lambda_{t,r}(g_t)) = d_r[\psi_{s,r}(\varphi_{t,s}(g_t))]$ for suitable $r\in R$ and $s\in S$, where d_r is the natural homomorphism of D_r in D^{∞}.

On the other hand, we have $\varphi^{\infty}(g) = f_s(\varphi_{t,s}(g_t))$ and therefore, in view of the definition of homomorphism ψ^{∞}, we obtain $\psi^{\infty}(\varphi^{\infty}(g)) = d_r[\psi_{s,r}(\varphi_{t,s}(g_t))]$. Thus $\lambda^{\infty}(g) = \psi^{\infty}(\varphi^{\infty}(g))$, which ends the proof.

Let $\{G_t, h_{t,t'}\}_{t\in T}$ be a direct system of groups over the directed set

T, $\sim$ — an equivalence relation in T satisfying conditions (a) and (b). Moreover, suppose that

(d) for any $t, t' \in T$, if $t \sim t'$, then there exists an element $t'' \in T$ such that $t \leqslant t''$, $t' \leqslant t''$ and $t \sim t''$.

In this situation every equivalence class $\tilde{t}$ is again a directed set and so we can take into consideration the partial direct system $\{G_{t'}, h_{t',t''}\}_{t' \in \tilde{t}}$ determined by $\tilde{t}$.

For any $\tilde{t}, \tilde{s}, \tilde{r} \in \tilde{T}$ such that $\tilde{t} \leqslant \tilde{s} \leqslant \tilde{r}$ the families of homomorphisms $\{h_{t',s'}\}, \{h_{s',r'}\}$ and $\{h_{t',r'}\}$ of $\{G_{t'}, h_{t',t''}\}_{t' \in \tilde{t}}$ in $\{G_{s'}, h_{s',s''}\}_{s' \in \tilde{s}}$, of $\{G_{s'}, h_{s',s''}\}_{s' \in \tilde{s}}$ in $\{G_{r'}, h_{r',r''}\}_{r' \in \tilde{r}}$ and of $\{G_{t'}, h_{t',t''}\}_{t' \in \tilde{t}}$ in $\{G_{r'}, h_{r',r''}\}_{r' \in \tilde{r}}$ satisfy conditions of Corollary 1.

Denote by $G_{\tilde{t}}^{\infty}$ the direct limit of the partial system $\{G_{t'}, h_{t',t''}\}_{t' \in \tilde{t}}$ and, for $\tilde{t} \leqslant \tilde{s}$, let $h_{\tilde{t},\tilde{s}}^{\infty}$ be the homomorphism of $G_{\tilde{t}}^{\infty}$ in $G_{\tilde{s}}^{\infty}$ as defined in the proof of theorem 1.

The following statement follows immediately from Corollary 1:

COROLLARY 2. *The family of groups $G_{\tilde{t}}^{\infty}$ together with the homomorphisms $h_{\tilde{t},\tilde{s}}^{\infty}$ form a direct system over $\tilde{T}$.*

Remark. If $\{G_t, h_{t,t'}\}_{t \in T}$ is a regular system, $\sim$ — a congruence relation in the semilattice T, then the derived system $\{G_{\tilde{t}}^{\infty}, h_{\tilde{t},\tilde{t}'}^{\infty}\}_{\tilde{t} \in \tilde{T}}$ is again regular.

§ 2. Commutative regular semigroups. A set G in which a binary associative operation is defined (called *addition* and denoted by $+$) is called a *semigroup*.

A semigroup G is called *commutative* if the operation $+$ is commutative, and it is said to be *regular* if for any $g \in G$ there exists an element $a \in G$ such that $g+a+g = g$ (6).

A subset H of the semigroup G is called a *subsemigroup* if $g_1 + g_2 \in H$ for any $g_1, g_2 \in H$.

We shall call *zero* every element x of the semigroup G such that $x+x = x$. The set of all zeros of G will be denoted by G_0.

A subsemigroup of a regular semigroup which is again regular will be called a *regular subsemigroup*.

If G is a commutative regular semigroup, then G_0 is, of course, a regular subsemigroup.

(6) Regular commutative semigroups have been studied in [3]. We have used there multiplicative notation for the semigroup operation. In this paper additive notation will be more convenient on account of similarity of the theory of commutative regular semigroups to that of Abelian groups. Just for that reason we shall modify the terminology and some notation, for example the direction of the partially ordering relation introduced in [3]. A larger class of semigroups has been studied by Clifford [1a].

A regular subsemigroup of a commutative regular semigroup G is said to be a *characteristic subsemigroup* if it contains the zero-subsemigroup G_0 of G.

We shall use later on in this paper four results, listed below, which have been proved in [3] or which are immediate consequences of theorems proved in [3]. In this list G always denotes a commutative regular semigroup.

(I) For any $g \in G$ there exists exactly one element $O(g) \in G_0$ and one element $-g \in G$ such that $g+O(g) = g$, $g+(-g) = O(g)$, $(-g)+O(g) = -g$, $O(-g) = O(g)$.

(II) For any $g_1, g_2 \in G$ we have $O(g_1+g_2) = O(g_1)+O(g_2)$, $-(g_1+g_2) = (-g_1)+(-g_2)$. If $g \in G_0$, then $O(g) = g$.

(III) For any $x \in G_0$ let G_x be the set of all $g \in G$ such that $O(g) = x$. G_x is an Abelian group with respect to the operation $+$, x is the zero of G_x, $-g$ the inverse of g. If $x \neq y$, then the groups G_x and G_y are disjoint. $G = \bigcup_{x \in G_0} G_x$.

(IV) For arbitrary $x, y \in G_0$ let $x \leqslant y$ if and only if $x+y = y$. The relation $\leqslant$ is a partially ordering relation and G_0 is a semilattice with respect to $\leqslant$, $x+y$ being the smallest upper bound of x and y (7).

THEOREM 2. *Let $x, y \in G_0$ and $x \leqslant y$. Define a function $h_{x,y}$ in G_x by the following formula*:

$$h_{x,y}(g) = g+y \quad \textit{for any} \quad g \in G_x.$$

Then

1° *$h_{x,y}$ is a homomorphism of G_x in G_y,*

2° *for any $x \leqslant y \leqslant z$ the diagram*

$$\begin{array}{ccc} G_x & \xrightarrow{\;h_{x,y}\;} & G_y \\ & h_{x,z} \searrow \quad \swarrow h_{y,z} & \\ & G_z & \end{array}$$

is commutative.

Proof. 1° In view of $x \leqslant y$, (II) implies $O(g+y) = O(g)+O(y) = x+y = y$, whence $g+y \in G_y$.

By (II), for any $g_1, g_2 \in G_x$ we have $h_{x,y}(g_1+g_2) = (g_1+g_2)+y = (g_1+y)+(g_2+y) = h_{x,y}(g_1)+h_{x,y}(g_2)$.

(7) In [3], $O(g)$ was denoted by $\alpha(g)$, and $-g$ by $\bar{g}$ (df. S5 and df. S1). (I) follows from theorem S2 and theorem S4 in [3]. Similarly, (II) is the form of theorem S7 and theorem S8. (III) follows from theorem 3 and theorem 4. Relation $\leqslant$ (with inverse direction) was introduced in [3] in the whole semigroup G (df. S4). (IV) follows from the remark after theorem S10.

2° If $x \leqslant y \leqslant z$, then we have $h_{y,z}(h_{x,y}(g)) = h_{y,z}(g+y) = (g+y)+z = g+z = h_{x,z}(g)$.

COROLLARY 3. *For any $g_1 \in G_x$ and $g_2 \in G_y$ we have* $h_{x,x+y}(g_1)+h_{y,x+y}(g_2) = g_1+g_2$.

Proof. $h_{x,x+y}(g_1)+h_{y,x+y}(g_2) = g_1+(x+y)+g_2+(x+y) = (g_1+g_2)+(x+y)$.

By (II) we have $O(g_1+g_2) = O(g_1)+O(g_2) = x+y$, whence, by (I), $(g_1+g_2)+(x+y) = g_1+g_2$, which ends the proof.

Remark. In view of theorems (III), (IV) and theorem 2 the family of groups G_x together with homomorphisms $h_{x,y}$ form a regular system of groups over the semilattice G_0. This system will be said to be *associated with the commutative regular semigroup G*.

THEOREM 3. *Commutative regular semigroups G and F are isomorphic if and only if their associated direct systems $\{G_x, h_{x,x'}\}_{x \in G_0}$ and $\{F_y, f_{y,y'}\}_{y \in F_0}$ are isomorphic.*

Proof. Let φ be an isomorphism of G onto F, let φ_0 be the restriction of φ to G_0, φ_x — restriction of φ to G_x. It is easy to see that φ_0 is a similarity-function of G_0 and F_0, φ_x — an isomorphism of G_x onto $F_{\varphi(x)}$. Let $x, x' \in G_0$, $x \leqslant x'$, $\varphi(x) = y$, $\varphi(x') = y'$. For any $g \in G_x$ we have $\varphi_{x'}(h_{x,x'}(g)) = \varphi(g+x') = \varphi(g)+\varphi(x') = \varphi(g)+y = f_{y,y'}(\varphi(g)) = f_{y,y'}(\varphi_x(g))$, thus φ_0 and the family $\{\varphi_x\}_{x \in G_0}$ establishes an isomorphism of the associated systems of G and F.

On the other hand, suppose that associated systems $\{G_x, h_{x,x'}\}_{x \in G_0}$ and $\{F_y, f_{y,y'}\}_{y \in F_0}$ of G and F are isomorphic, φ_0 and $\{\varphi_x\}_{x \in G_0}$ being the corresponding mappings. Then the function φ defined for $g \in G$ by $\varphi(g) = \varphi_x(g)$ if $g \in G_x$ is an isomorphism of G onto F.

In fact, φ is one-to-one and, for any $g_1 \in G_{x_1}$ and $g_2 \in G_{x_2}$, if we write $x_1+x_2 = x$, $\varphi_0(x_1) = y_1$, $\varphi_0(x_2) = y_2$, $\varphi(x) = y$, then in virtue of Corollary 3, of the commutativity of suitable diagrams and of the fact that φ_x is an isomorphism, we get

$$\begin{aligned}\varphi(g_1+g_2) &= \varphi\big(h_{x_1,x}(g_1)+h_{x_2,x}(g_2)\big) = \varphi_x\big(h_{x_1,x}(g_1)+h_{x_2,x}(g_2)\big)\\ &= \varphi_x\big(h_{x_1,x}(g_1)\big)+\varphi_x\big(h_{x_2,x}(g_2)\big) = f_{y_1,y}\big(\varphi_{x_1}(g_1)\big)+f_{y_2,y}\big(\varphi_{x_2}(g_2)\big)\\ &= \varphi_{x_1}(g_1)+\varphi_{x_2}(g_2) = \varphi(g_1)+\varphi(g_2).\end{aligned}$$

Now we shall prove that every regular system of groups is the associated system of a suitable commutative regular semigroup.

THEOREM 4. *Let $\{G_x, h_{x,y}\}_{x \in G_0}$ be a regular system of Abelian groups over a semilattice G_0. The set $G = \bigcup_{x \in G_0} G_x$ is a commutative regular semigroup* *with respect to operation + defined as follows: for any $g_1, g_2 \in G$ if $g_1 \in G_x$ and $g_2 \in G_y$, we set*

$$g_1 + g_2 = h_{x,x+y}(g_1) + h_{y,x+y}(g_2)\ ([8]).$$

Moreover, if we identify index x with the zero of the group G_x, then the system $\{G_x, h_{x,y}\}_{x \in G_0}$ will be identical with the associated system of G.

Proof. From the disjointness of the groups G_x it follows at once that any $g \in G$ belongs exactly to one group G_x, so the operation + in G is well defined.

Commutativity of the operation + in G follows from commutativity of the group operation in the groups G_x.

To prove the associativity observe that in virtue of the commutativity of suitable diagrams formed by the homomorphisms $h_{x,y}$ we obtain for any $g_1 \in G_x$, $g_2 \in G_y$ and $g_3 \in G_z$ the equations $(g_1+g_2)+g_3 = \big(h_{x,x+y+z}(g_1) + h_{y,x+y+z}(g_2)\big) + h_{z,x+y+z}(g_3)$ and $g_1 + (g_2+g_3) = h_{x,x+y+z}(g_1) + \big(h_{y,x+y+z}(g_2) + h_{z,x+y+z}(g_3)\big)$, and thus the equality $(g_1+g_2)+g_3 = g_1+(g_2+g_3)$ immediately follows from the associativity of the group operation in G_{x+y+z}.

Observe that since $h_{x,x}$ is the identity isomorphism of G_x, the operation + defined in G coincides for elements of G_x with the group operation in G_x. So for any $g \in G_x$ we have $g+(-g)+g = g$, where $-g$ is the inverse of g in G_x, thus G is regular.

If we identify index x with the zero of G_x (the zero-subsemigroup of G is then identified with the semilattice G_0), then the order relation $\leqslant$ introduced in the zero-subsemigroup of G in the manner described in theorem (IV) will be identical with the previous order relation in G_0.

To end the proof that $\{G_x, h_{x,y}\}_{x \in G_0}$ is the associated system of G we must show that $h_{x,y}(g) = g+y$ for any $x \leqslant y$, $(x, y \in G_0)$ and for any $g \in G_x$.

In fact, it immediately follows from the definition of the operation + in G that $g+y = h_{x,x+y}(g) + h_{y,y+y}(y) = h_{x,y}(g) + h_{y,y}(y) = h_{x,y}(g) + y = h_{x,y}(g)$.

In view of theorems 2, 3 and 4 there is a one-to-one correspondence between commutative regular semigroups and regular systems of Abelian groups. Every commutative regular semigroup may be identified with its associated system. This identification will be used to determining all homomorphic images of a given commutative regular semigroup.

THEOREM 5. *Let G be a commutative regular semigroup, H — a subset of G and let $H_x = H \cap G_x$. H is a characteristic subsemigroup of G if and only if $\{H_x, h_{x,y}\}_{x \in G_0}$ is a subsystem of $\{G_x, h_{x,y}\}_{x \in G_0}$.*

([8]) We use here the same symbol + for the group operation in any group G_x and for the smallest upper bound in the semilattice G_0.

Proof. 1) Let H be a characteristic subsemigroup of G. It follows immediately from the regularity of H that H_x is a subgroup of G_x for every $x \in G_0$. Moreover, $h_{x,y}(g) = g + y \in H_y$ for any $g \in H_x$ and an arbitrary $y \in G_0$ such that $x \leqslant y$, since $G_0 \subset H$. Thus $\{H_x, h_{x,y}\}_{x \in G_0}$ is a subsystem of $\{G_x, h_{x,y}\}_{x \in G_0}$.

2) Now, let $\{H_x, h_{x,y}\}_{x \in G_0}$ be a subsystem of the associated system of G. We must prove that $H = \bigcup_{x \in G_0} H_x$ is a characteristic subsemigroup of G.

H_x is a group, whence $x \in H_x$. Thus H contains the zero-subsemigroup G_0 of G. If $g_1 \in H_x$ and $g_2 \in H_y$, then, by Corollary 3, we have $g_1 + g_2 = h_{x,x+y}(g_1) + h_{y,x+y}(g_2) \in H_{x+y}$, thus H is a subsemigroup. The regularity of H follows at once from the fact that all H_x are groups.

Now we shall study homomorphic images of a given commutative regular semigroup. It is easy to see that a homomorphic image of a commutative regular semigroup is again commutative and regular.

Let φ be a homomorphism of the commutative regular semigroup G onto F. The set of all $g \in G$ such that $\varphi(g) \in F_0$ (i. e., $\varphi(g)$ is a zero in F) will be called the *kernel* of the homomorphism φ and will be denoted by $\mathrm{Ker}\varphi$.

THEOREM 6. *For every homomorphism φ of an arbitrary commutative regular semigroup G, $\mathrm{Ker}\varphi$ is a characteristic subsemigroup of G.*

Proof. If $x \in G_0$, then $\varphi(x) = \varphi(x+x) = \varphi(x) + \varphi(x)$, whence $\varphi(x) \in F_0$, thus $x \in \mathrm{Ker}\varphi$.

If $g_1, g_2 \in \mathrm{Ker}\varphi$, then $\varphi(g_1), \varphi(g_2) \in F_0$. Hence $\varphi(g_1 + g_2) = \varphi(g_1) + \varphi(g_2) \in F_0$ and $g_1 + g_2 \in \mathrm{Ker}\varphi$.

It is evident that $\mathrm{Ker}\varphi \cap G_x$ is a subgroup of G_x and therefore it follows at once that $\mathrm{Ker}\varphi$ is regular. This ends the proof.

Let φ be a homomorphism of the commutative regular semigroup G onto F. We shall define a relation $\sim$ in G_0 as follows: for any $x, x' \in G_0$, let $x \sim x'$ if and only if $\varphi(x) = \varphi(x')$. It is evident that

1° the relation $\sim$ is a congruence relation in the semilattice G_0,

2° the set $\tilde{G}_0$ of equivalence classes with the induced order is similar to F_0.

Thus every equivalence class $\tilde{x}$ $(x \in G_0)$ is again a semilattice and so we may consider the partial system of $\{G_x, h_{x,y}\}_{x \in G_0}$ determined by $\tilde{x}$.

On the other hand, since $\mathrm{Ker}\varphi$ is a characteristic subgroup, it determines (see theorem 5) a subsystem $\{K_x, h_{x,y}\}_{x \in G_0}$ $(K_x = \mathrm{Ker}\varphi \cap G_x)$ of $\{G_x, h_{x,y}\}_{x \in G_0}$.

The partial system $\{K_z, h_{z,z'}\}_{z \in \tilde{x}}$ is of course a subsystem of the partial system $\{G_z, h_{z,z'}\}_{z \in \tilde{x}}$. Thus we can form the factor system $\{G_z/K_z, \bar{h}_{z,z'}\}_{z \in \tilde{x}}$. Let $\bar{G}_{\tilde{x}}^{\infty}$ be the direct limit of the last system.

LEMMA 1. *The group $\bar{G}_{\tilde{x}}^{\infty}$ is isomorphic with F_y, where $y = \varphi(x)$.*

Proof. It is easy to see that the family of subgroups $\{\varphi(G_z)\}_{z\in\tilde{x}}$ of the group F_y ($\varphi(G_z)$ is the image of the group G_z under homomorphism φ) satisfies the conditions of theorem (V). Thus F_y is isomorphic with the direct limit of the direct system composed of the groups $\varphi(G_z)$ and embedding-homomorphisms $i_{z,z'}$.

On the other hand, let $\overline{\varphi}_z$ be the natural isomorphism of G_z/K_z onto $\varphi(G_z)$ (i. e., the isomorphism which maps the K_z-residue class of $g\in G_z$ into $\varphi(g)$).

It is easy to see that the diagram

$$\begin{array}{ccc} G_z/K_z & \xrightarrow{\overline{h}_{z,z'}} & G_{z'}/K_{z'} \\ \overline{\varphi}_z \downarrow & & \downarrow \overline{\varphi}_{z'} \\ \varphi(G_z) & \xrightarrow[i_{z,z'}]{} & \varphi(G_{z'}) \end{array}$$

is commutative for any $z, z'\in\tilde{x}$ such that $z\leqslant z'$.

Hence the systems $\{G_z/K_z, \overline{h}_{z,z'}\}_{z\in\tilde{x}}$ and $\{\varphi(G_z), i_{z,z'}\}_{z\in\tilde{x}}$ are isomorphic and therefore it follows that their direct limits are isomorphic too.

Remark. The natural isomorphism of $\overline{G}_{\tilde{x}}^{\infty}$ onto F_y may be defined as follows: for any $g\in\overline{G}_{\tilde{x}}^{\infty}$ we take an arbitrary $\bar{g}_z\in G_z/K_z$ ($z\in\tilde{x}$) such that $\overline{h}_z(\bar{g}_z) = g$ ($\overline{h}_z$ is the natural homomorphism of G_z/K_z in $\overline{G}_{\tilde{x}}^{\infty}$) and we set $\overline{\varphi}_{\tilde{x}}^{\infty}(g) = \overline{\varphi}_z(\bar{g}_z)$.

Now, all the direct limits $\overline{G}_{\tilde{x}}^{\infty}$ of partial direct systems determined by equivalence classes $\tilde{x}$ together with homomorphisms $\overline{h}_{\tilde{x},\tilde{x}'}^{\infty}$ described in the proof of theorem 1 form again a direct system with $\tilde{G}_0$ as its index set (see Corollary 2).

It is not hard to verify the following

LEMMA 2. *Direct systems* $\{\overline{G}_{\tilde{x}}^{\infty}, \overline{h}_{\tilde{x},\tilde{x}'}^{\infty}\}_{\tilde{x}\in\tilde{G}_0}$ *and* $\{F_y, f_{y,y'}\}_{y\in F_0}$ *are isomorphic. The isomorphism is established by the similarity-mapping* $\overline{\varphi}_0$ *of* $\tilde{G}_0$ *onto* F_0 *which maps the class* $\tilde{x}\in\tilde{G}_0$ *onto* $\varphi(x)$ *and the family of isomorphisms* $\{\overline{\varphi}_{\tilde{x}}^{\infty}\}_{\tilde{x}\in\tilde{G}_0}$.

The following theorem is an immediate consequence of lemma 2 and theorem 3:

THEOREM 7. *The commutative regular semigroup determined by the system* $\{\overline{G}_{\tilde{x}}^{\infty}, \overline{h}_{\tilde{x},\tilde{x}'}^{\infty}\}_{\tilde{x}\in\tilde{G}_0}$ *is isomorphic with the semigroup* F.

Let H be a characteristic subgroup of a commutative regular semigroup G, $\{G_x, h_{x,y}\}_{x\in G_0}$ and $\{H_x, h_{x,y}\}_{x\in G_0}$ — their associated systems and let $\sim$ be a congruence relation in the zero-subgroup G_0 of G. We can construct the factor system $\{G_x/H_x, \overline{h}_{x,y}\}_{x\in G_0}$ and further on, in the manner described in Corollary 2, the direct system $\{\overline{G}_{\tilde{x}}^{\infty}, \overline{h}_{\tilde{x},\tilde{y}}^{\infty}\}_{\tilde{x}\in\tilde{G}_0}$. The last system is regular, since $\tilde{G}_0$ is a semilattice. Thus it determines a commutative regular semigroup which we shall call the *factor semigroup* of G with respect to the characteristic subsemigroup H and the congruence relation $\sim$. We shall denote it by $G/(H, \sim)$.

We are now in a position to formulate a theorem which is the inverse of theorem 7:

THEOREM 8. *For every characteristic subsemigroup H of a commutative regular semigroup G and every congruence relation $\sim$ in the zero-subsemigroup G_0 of G the factor semigroup $G/(H, \sim)$ is a homomorphic image of G.*

Proof. Let g be an arbitrary element of G. There exists an element $x \in G_0$ such that $g \in G_x$. We shall denote by $\bar{g}$ the H_x-residue class of g. Let $\varphi(g) = \bar{h}_x(\bar{g})$, where $\bar{h}_x$ is the natural homomorphism of G_x/H_x in $\bar{G}^{\infty}_{\tilde{x}}$.

We shall prove that φ is the required homomorphism of G onto $G/(H, \sim)$.

Let $g_1 \in G_{x_1}, g_2 \in G_{x_2}, \bar{g}_1 = g_1 + H_{x_1}, \bar{g}_2 = g_2 + H_{x_2}$ and let $x_1 + x_2 = x$. Of course, we have $\overline{g_1 + g_2} = \bar{g}_1 + \bar{g}_2 \in G_x/H_x$ and $\tilde{x}_1 + \tilde{x}_2 = \tilde{x}$.

It follows from theorem 1 that diagrams

$$\begin{array}{ccc} G_{x_i}/H_{x_i} & \xrightarrow{\ \bar{h}_{x_i}\ } & \bar{G}^{\infty}_{\tilde{x}_i} \\ \bar{h}_{x_i, x} \downarrow & & \downarrow \bar{h}^{\infty}_{\tilde{x}_i, \tilde{x}} \\ G_x/H_x & \xrightarrow[\ \bar{h}_x\ ]{} & \bar{G}^{\infty}_{\tilde{x}} \end{array} \tag{9}$$

$(i = 1, 2)$ are commutative, whence by Corollary 3 and the definition of operation $+$ in $G/(H, \sim)$ (see theorem 3) we obtain: $\varphi(g_1) + \varphi(g_2) = = \bar{h}_{x_1}(\bar{g}_1) + \bar{h}_{x_2}(\bar{g}_2) = \bar{h}^{\infty}_{\tilde{x}_1, \tilde{x}}(\bar{h}_{x_1}(\bar{g}_1)) + \bar{h}^{\infty}_{\tilde{x}_2, \tilde{x}}(\bar{h}_{x_2}(\bar{g}_2)) = \bar{h}_x(\bar{h}_{x_1, x}(g_1)) + \bar{h}_x(\bar{h}_{x_2, x}(\bar{g}_2)) = \bar{h}_x(\bar{h}_{x_1, x}(\bar{g}_1) + \bar{h}_{x_2, x}(\bar{g}_2)) = \bar{h}_x(\bar{g}_1 + \bar{g}_2) = \varphi(g_1 + g_2)$.

It is evident that the image of G under homomorphism φ is the whole semigroup $G/(H, \sim)$.

It follows immediately from theorem 7 that if φ is a homomorphism of a commutative regular semigroup G onto F, $H = \mathrm{Ker}\varphi$, $\sim$ — the congruence relation in G_0 determined by φ (i. e. $x \sim x'$ if and only if $\varphi(x) = \varphi(x')$), then the semigroup F is isomorphic with the factor semigroup $G/(H, \sim)$.

Thus we have proved

COROLLARY 4. *Factor semigroups of a given commutative regular semigroup G with respect to characteristic subsemigroups and congruence relations of the zero-subsemigroup G_0 are the only homomorphic images of G.*

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