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A GENERALIZATION OF THE LOOMIS-SIKORSKI THEOREM

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1. Preliminaries and notation. We shall consider some distributive lattices with zero-element denoted by 0 and unit-element denoted by 1 — and their topological representations.

The lattice operations will be denoted by $\cup$ and $\cap$, and will be called *join* and *meet*, respectively. Sometimes, however, it is convenient, in Post algebras, to write xy instead of $x \cap y$. The complement of x, if it exists in a lattice, will be denoted by $-x$. Instead of $x \cap -y$ we shall write $x-y$. For a partly ordering relation in a lattice the symbol $\leqslant$ is provided; for the set-inclusion the symbol $\subset$ is used, as usually.

Let L be a fixed distributive lattice with 0 and 1. The set B of all complemented elements of L, if considered as a sublattice of L, is a Boolean algebra. If there exists an ascending sequence

$$0 = e_0 \leqslant e_1 \leqslant \ldots \leqslant e_{n-1} = 1 \tag{1}$$

of different elements of L, $n \geqslant 2$, such that every $x \in L$ can be written in the form

$$x = b_1 e_1 \cup b_2 e_2 \cup \ldots \cup b_{n-1} e_{n-1}, \tag{2}$$

where $b_1, \ldots, b_{n-1} \in B$ and $e_1, \ldots, e_{n-2}$ do not belong to the Boolean algebra B, then L is said to be a *P_0-lattice of order n* ([6], p. 194).

Representation (2) is said to be *monotonic* if $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_{n-1}$.

The P_0-lattice being determined by the sequence $e_0, e_1, \ldots, e_{n-1}$ and by the Boolean algebra B, it is convenient to write

$$L = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle.$$

By a *Post algebra* (of order n) we mean a P_0-lattice $P = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$ satisfying an additional condition (see [6], p. 198)

(3) if $a \in B$ and $ae_i \leqslant e_{i-1}$ for some i, then $a = 0$.

For other equivalent definitions see Rosenbloom [4], Epstein [1] and Traczyk [6].

A P_0-lattice (a Post algebra) of subsets of a set with set-theoretical union and intersection as lattice operations join and meet, respectively, will be called a *P_0-lattice of sets* (a *Post field*). Consequently a field will be called a *Boolean field* in this paper.

If $\mathfrak{X}$ is a topological space and for a set $\mathfrak{X}_0 \subset \mathfrak{X}$ the relative topology is assumed, then $\mathfrak{X}_0$ is said to be a *subspace* of the space $\mathfrak{X}$ (Kelley [2], p. 51).

The abbrevations: Post $\mathfrak{m}$-algebra, Post $\mathfrak{m}$-field, $\mathfrak{m}$-ideal will be used for: $\mathfrak{m}$-complete Post algebra, $\mathfrak{m}$-complete Post field, $\mathfrak{m}$-complete ideal, where $\mathfrak{m}$ denotes an infinite cardinal.

Now let us fix a Post algebra $P = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$. An ideal Δ of P is said to be *of order* i, if $e_{i-1} \in \Delta$ and $e_i \notin \Delta$. Let $\mathfrak{X}_0$ be the set of all prime ideals of the Boolean algebra B and $\mathfrak{X}_i$ the set of all prime ideals of P of order i $(i = 1, 2, \ldots, n-1)$. Then the following statement is true (see [6], p. 200-202):

1 (i) *If Δ_i is a prime ideal of P of order i, then $\Delta_0 = \Delta_i \cap B$ is a prime ideal of the Boolean algebra B and the mapping Φ_i: $\mathfrak{X}_i \to \mathfrak{X}_0$ defined by the formula*

$$(*) \qquad \Phi_i(\Delta_i) = \Delta_i \cap B$$

is one-to-one and the range of Φ_i is the whole space $\mathfrak{X}_0$.

2. Stone space of a Post algebra. For every $a \in B$ let $h_0(a) \subset \mathfrak{X}_0$ be the set of all prime ideals Δ_0 of B such that $a \notin \Delta_0$. Let $\boldsymbol{F}_0 = \{h_0(a): a \in B\}$. It is well known that h_0 is an isomorphism of B onto the field $\boldsymbol{F}_0$ of subsets of $\mathfrak{X}_0$ and that $\mathfrak{X}_0$ becomes a compact totally disconnected space if we assume $\boldsymbol{F}_0$ to be the open basis for the topology for it. $\boldsymbol{F}_0$ coincides with the field of all both open and closed subsets of $\mathfrak{X}_0$.

The space $\mathfrak{X}_0$ is the Stone space of the Boolean algebra B as well as each its homeomorph (see e. g. [5], p. 21-23). Then each $\mathfrak{X}_i$ (see 1 (i)) becomes a Stone space of B, too, if we assume $\Phi_i^{-1}(\boldsymbol{F}_0) = \boldsymbol{F}_i$ to be the open basis for the topology for it.

Let us put

$$\mathfrak{X} = \mathfrak{X}_1 \cup \mathfrak{X}_2 \cup \ldots \cup \mathfrak{X}_{n-1},$$

$$h(a) = \Phi_1^{-1}h_0(a) \cup \ldots \cup \Phi_{n-1}^{-1}h_0(a) \quad \text{for every} \quad a \in B,$$

$$\boldsymbol{F} = \{h(a): a \in B\}.$$

2 (i) *The class $\boldsymbol{F}$ is a Boolean field of subsets of $\mathfrak{X}$ and h is an isomorphism of the Boolean algebra B onto $\boldsymbol{F}$.*

Proof. We observe first that the mapping $\Phi_i^{-1}h_0$ is a homomorphism of B into the field of all subsets of the set $\mathfrak{X}_i$. But Φ_i^{-1} is an isomorphism of the field of all subsets of $\mathfrak{X}_0$ onto the field of all subsets of $\mathfrak{X}_i$, by virtue of 1 (i). Therefore the superposition $\Phi_i^{-1}h_0$ is an isomorphism of the Boolean algebra B into the field of all subsets of $\mathfrak{X}_i$.

Now, if we do not forget that $\mathfrak{X}_i\mathfrak{X}_j = 0$ for $i \neq j$, there will be no troubles in proving that h is an isomorphism of B into the field of all subsets of $\mathfrak{X}$. The details will be omitted. Since $\boldsymbol{F} = h(B)$, it is a Boolean field isomorphic to the Boolean algebra B.

2 (ii) *Let $\boldsymbol{F}$ be a basis for the topology for $\mathfrak{X}$. Then $\mathfrak{X}$ is a compact topological space, the totally disconnected space $\mathfrak{X}_i$ $(i = 1, \ldots, n-1)$ is a subspace of $\mathfrak{X}$ and it is a set of the second category at every point of $\mathfrak{X}$.*

Proof. First let us see that for every set $A_i \in \boldsymbol{F}_i$ there exists an $a \in B$ such that $A_i = \Phi_i^{-1}h_0(a) = h(a)\mathfrak{X}_i$. This means that $\mathfrak{X}_i$ is a subspace of $\mathfrak{X}$.

Therefore for every non-empty open set $G \subset \mathfrak{X}$ the non-empty intersection $G\mathfrak{X}_i$ is open in the compact Hausdorff space $\mathfrak{X}_i$ and thus not of the first category (Čech [1]).

The compactness of the whole space $\mathfrak{X}$ follows easily from the compactness of $\mathfrak{X}_i$, $i = 1, \ldots, n-1$.

2 (iii) *The Boolean field $\boldsymbol{F}$ coincides with the class of all both closed and open subsets of the space $\mathfrak{X}$.*

The easy proof is omitted.

Let us write

$$E_0 = 0, \quad E_1 = \mathfrak{X}_1,$$

$$E_2 = \mathfrak{X}_1 \cup \mathfrak{X}_2, \quad \ldots, \quad E_{n-1} = \mathfrak{X}_1 \cup \mathfrak{X}_2 \cup \ldots \cup \mathfrak{X}_{n-1} = \mathfrak{X}$$

and let $\boldsymbol{R}$ be the class of all subsets $X \subset \mathfrak{X}$ of the form

$$X = A_1E_1 \cup A_2E_2 \cup \ldots \cup A_{n-1},$$

where $A_i \in \boldsymbol{F}$, $i = 1, \ldots, n-1$.

2 (iv) *$\boldsymbol{R}$ is a Post field of subsets of $\mathfrak{X}$ isomorphic to the Post algebra P.*

Proof. We infer from the definition that the class $\boldsymbol{R}$ is a P_0-lattice of sets, namely

$$\boldsymbol{R} = \langle E_0, E_1, \ldots, E_{n-1}; \boldsymbol{F}\rangle.$$

If $AE_i \subset E_{i-1}$ for some $A \in \boldsymbol{F}$ and some i, then $A\mathfrak{X}_i = 0$, whence $A = 0$, the set $\mathfrak{X}_i$ being dense in $\mathfrak{X}$ (see 2 (ii)).

Consequently, by (3), the P_0-lattice $\boldsymbol{R}$ is a Post field.

The isomorphism h of B onto $\boldsymbol{F}$ can be extended to the isomorphism of P onto $\boldsymbol{R}$ (see [6], p. 202), q. e. d.

A compact topological space $\mathfrak{X}$ is said to be the *Stone space of a Post algebra* $P = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$, provided P is isomorphic to the Post field $\boldsymbol{R} = \langle E_0, E_1, \ldots, E_{n-1}; \boldsymbol{F}\rangle$ of subsets of $\mathfrak{X}$, where $\boldsymbol{F}$ is the Boolean field of all both closed and open sets and $E_i - E_{i-1}$ $(i = 1, \ldots, n-1)$ is a compact totally disconnected subspace of $\mathfrak{X}$, and E_0 is the empty set.

2 (v) *If $\mathfrak{X}$ is a Stone space of the Post algebra P, then the subspace $\mathfrak{X}_i = E_i - E_{i-1}$ $(i = 1, \ldots, n-1)$ is a Stone space of the Boolean algebra B. All Stone spaces of P are homeomorphic.*

Proof. Let $\mathfrak{X}$ be a Stone space of P and let

$$\boldsymbol{R} = \langle E_0, E_1, \ldots, E_{n-1}; \boldsymbol{F} \rangle$$

be a Post field of subsets of $\mathfrak{X}$ isomorphic to P.

The formula

$$h(A) = A\mathfrak{X}_i \quad \text{for} \quad A \in \boldsymbol{F}$$

defines a homeomorphism of $\boldsymbol{F}$ into the field of all subsets of $\mathfrak{X}_i$. For every $A \in \boldsymbol{F}$

$$A\mathfrak{X}_i = 0 \quad \text{implies} \quad A = 0,$$

because, if not, then AE_i would be contained in E_{i-1} for a non-zero element of $\boldsymbol{F}$ and thus $\boldsymbol{R}$ would not be a Post field (see (3)).

Consequently h is an isomorphism of the Boolean field $\boldsymbol{F}$ into the field of all subsets of $\mathfrak{X}_i$. Therefore $\boldsymbol{F}_i = h(\boldsymbol{F})$ is a Boolean field of all both closed and open subsets of the subspace $\mathfrak{X}_i$, isomorphic to the Boolean algebra B. It follows that the totally disconnected space $\mathfrak{X}_i$ is the Stone space of the Boolean algebra B $(i = 1, \ldots, n-1)$.

The second part of the theorem is obvious.

2 (vi) *If there exists a finite sequence $\mathfrak{X}_1, \ldots, \mathfrak{X}_{n-1}$ of disjoint subsets of a compact topological space $\mathfrak{X}$ such that*

(a_1) $\mathfrak{X} = \mathfrak{X}_1 \cup \mathfrak{X}_2 \cup \ldots \cup \mathfrak{X}_{n-1}$,

(a_2) *$\mathfrak{X}_1, \ldots, \mathfrak{X}_{n-1}$ are compact totally disconnected subspaces of the space $\mathfrak{X}$,*

(a_3) *they are homeomorphic each to the other,*

then $\mathfrak{X}$ is a Stone space of a Post algebra.

The easy proof is omitted.

Let $\mathfrak{m}$ be an infinite cardinal number. A subset H of a topological space $\mathfrak{X}$ is said to be $\mathfrak{m}$-*closed* ($\mathfrak{m}$-*open*) if it is the intersection (the union) of at most $\mathfrak{m}$ sets both closed and open in $\mathfrak{X}$. A subset N of $\mathfrak{X}$ is said to be $\mathfrak{m}$-*nowhere dense* if it is a subset of a nowhere dense $\mathfrak{m}$-closed set. A subset A of $\mathfrak{X}$ is said to be of the $\mathfrak{m}$-*category* if it is the union of at most $\mathfrak{m}$ sets $\mathfrak{m}$-nowhere dense in $\mathfrak{X}$ (see, e. g., [5], p. 72-73).

2 (vii) *If $\mathfrak{X}$ is a compact space and $\mathfrak{X}_1, \ldots, \mathfrak{X}_{n-1}$ are compact totally disconnected subspaces of $\mathfrak{X}$, disjoint, homeomorphic each to the other, and such that $\mathfrak{X}_1 \cup \mathfrak{X}_2 \cup \ldots \cup \mathfrak{X}_{n-1} = \mathfrak{X}$, then*

(b_1) *a subset* H *of* $\mathfrak{X}$ *is* $\mathfrak{m}$*-closed* ($\mathfrak{m}$*-open*) *if and only if the set* $H\mathfrak{X}_i$ *is* $\mathfrak{m}$*-closed* ($\mathfrak{m}$*-open*) *in the subspace* $\mathfrak{X}_i$ *for* $i = 1, \ldots, n-1$;

(b_2) *a subset* H *of* $\mathfrak{X}$ *is* $\mathfrak{m}$*-nowhere dense in* $\mathfrak{X}$ *if and only if the set* $H\mathfrak{X}_i$ *is* $\mathfrak{m}$*-nowhere dense in* $\mathfrak{X}_i$ *for* $i = 1, \ldots, n-1$;

(b_3) *a subset* H *of* $\mathfrak{X}$ *is of the* $\mathfrak{m}$*-category in* $\mathfrak{X}$ *if and only if the set* $H\mathfrak{X}_i$ *is of the* $\mathfrak{m}$*-category in the subspace* $\mathfrak{X}_i$ *of* $\mathfrak{X}$ *for* $i = 1, \ldots, n-1$.

The easy proof is omitted.

3. $\mathfrak{m}$-representability of Post $\mathfrak{m}$-algebras. Let $L = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$ be a P_0-lattice and let Δ be an ideal of the Boolean algebra B with the property

(v) if $a \in B$ and $ae_i \leqslant e_{i-1}$ for some $i > 0$, then $a \in \Delta$.

If

$$x = a_1e_1 \cup a_2e_2 \cup \ldots \cup a_{n-1}e_{n-1},$$

$$y = b_1e_1 \cup b_2e_2 \cup \ldots \cup b_{n-1}e_{n-1}$$

are monotonic representations of elements x and y of L, then let us write

$$x \equiv y \text{ if and only if } b_i - a_i \cup a_i - b_i \in \Delta \text{ for } i = 1, 2, \ldots, n-1.$$

By the assumed property (v) of Δ, the relation $\equiv$ is transitive, reflexive and symmetrical ([6], p. 208). The abstract class of this relation, containing an element x of L, will be denoted by $[x]$. The set L/Δ of all abstract classes $[x]$, where x runs over L, becomes a Post algebra under the following definition of operations:

$$[x] \cup [y] = [x \cup y], \quad [x] \cap [y] = [x \cap y]$$

(see [6], p. 208). Such an algebra will be called a *quotient Post algebra.*

A Post $\mathfrak{m}$-algebra is said to be $\mathfrak{m}$*-representable* if it is isomorphic to a quotient Post algebra $\boldsymbol{R}_{\mathfrak{m}}/\Delta_{\mathfrak{m}}$, where $\boldsymbol{R}_{\mathfrak{m}} = \langle E_0, E_1, \ldots, E_{n-1}; \boldsymbol{F}_{\mathfrak{m}}\rangle$ is a P_0-lattice of sets, $\boldsymbol{F}_{\mathfrak{m}}$ is a Boolean $\mathfrak{m}$-field and $\Delta_{\mathfrak{m}}$ is an $\mathfrak{m}$-ideal of sets in $\boldsymbol{F}_{\mathfrak{m}}$.

Let us fix a Post $\mathfrak{m}$-algebra $P = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$ and an isomorphism h_0 of P into the field of all subsets of a Stone space $\mathfrak{X}$ of P. Let

$$h_0(P) = \boldsymbol{R} = \langle E_0, E_1, \ldots, E_{n-1}; \boldsymbol{F}\rangle$$

be a Post field of subsets of $\mathfrak{X}$ isomorphic to P. Then $\boldsymbol{F}$ is the Boolean field of all both open and closed subsets of $\mathfrak{X}$, isomorphic to the Boolean algebra B. Let $\boldsymbol{F}_{\mathfrak{m}}$ be the least Boolean $\mathfrak{m}$-field of subsets of $\mathfrak{X}$ containing $\boldsymbol{F}$ and let $\Delta_{\mathfrak{m}}$ be the $\mathfrak{m}$-ideal of all sets in $\boldsymbol{F}_{\mathfrak{m}}$ of the $\mathfrak{m}$-category.

3 (i) $\boldsymbol{F}_{\mathfrak{m}}$ *coincides with the class of all subsets of* $\mathfrak{X}$ *of the form*

$$H \cup N_1 - N_2, \quad \textit{where} \quad H \in \boldsymbol{F} \textit{ and } N_1, N_2 \in \Delta_{\mathfrak{m}}.$$

The proof of an analogous statement, where $\mathfrak{X}$ is a Stone space of a Boolean $\mathfrak{m}$-algebra ([5], p. 45), may be adopted here without any change in virtue of 2 (vii).

3 (ii) *Let the Boolean algebra B be $\mathfrak{m}$-representable. Then the ideal $\Delta_{\mathfrak{m}}$ has the property* (v) *in reference to the P_0-lattice of sets $\boldsymbol{R}_{\mathfrak{m}} = \langle E_0, E_1, \ldots, E_{n-1}; \boldsymbol{F}_{\mathfrak{m}}\rangle$, i. e., if $A \in \boldsymbol{F}_{\mathfrak{m}}$ and $AE_i \subset E_{i-1}$ for some $i > 0$, then $A \in \Delta_{\mathfrak{m}}$.*

Proof. By virtue of 3 (i)

$$A = H \cup N_1 - N_2, \quad \text{where} \quad H \in \boldsymbol{F} \text{ and } N_1, N_2 \in \Delta_{\mathfrak{m}}.$$

The Boolean algebra B being $\mathfrak{m}$-representable, no open non-empty subset of the Stone space of B is of the $\mathfrak{m}$-category (see e. g. [5], p. 96). By 2 (v) the subspace $\mathfrak{X}_i = E_i - E_{i-1}$ of $\mathfrak{X}$ is a Stone space of B. If $AE_i \subset E_{i-1}$, then $H\mathfrak{X}_i$ is of the $\mathfrak{m}$-category. But $H\mathfrak{X}_i$ is an open (and closed) subset of the space $\mathfrak{X}_i$. Consequently H is an empty set. Hence $A \in \Delta_{\mathfrak{m}}$, q. e. d.

The main result of the present paper is the following theorem:

3 (iii) *The Post $\mathfrak{m}$-algebra $P = \langle e_0, e_1, \ldots, e_{n-1}; B\rangle$ is $\mathfrak{m}$-representable if and only if the Boolean algebra B is $\mathfrak{m}$-representable.*

Proof. Since the Post algebra P is $\mathfrak{m}$-complete, the Boolean algebra B is $\mathfrak{m}$-complete as well (see [2]). P being $\mathfrak{m}$-representable, so is B, by definition.

We are going to prove the sufficiency of the theorem. By the hypothesis the mapping h defined by the formula

$$h(a) = [h_0(a)], \quad a \in B,$$

is an isomorphism of B onto $\boldsymbol{F}_{\mathfrak{m}}/\Delta_{\mathfrak{m}}$ (see, e. g., [5], p. 95). Let us consider once more the P_0-lattice $\boldsymbol{R}_{\mathfrak{m}} = \langle E_0, E_1, E_2, \ldots, E_{n-1}; \boldsymbol{F}_{\mathfrak{m}}\rangle$. In virtue of 3 (ii) there exists a quotient Post algebra $\boldsymbol{R}_{\mathfrak{m}}/\Delta_{\mathfrak{m}}$ and it is of the form

$$\langle [E_0], [E_1], \ldots, [E_{n-1}]; \boldsymbol{F}_{\mathfrak{m}}/\Delta_{\mathfrak{m}}\rangle.$$

The above defined isomorphism h may be extended in a natural way (see [6], p. 202) to an isomorphism h^* of P onto $\boldsymbol{R}_{\mathfrak{m}}/\Delta_{\mathfrak{m}}$.

In fact, if

$$(+) \qquad x_t = D_1(x_t)e_1 \cup \ldots \cup D_{n-1}(x_t)e_{n-1}$$

is a monotonic representation of x_t, $t \in T$, $\overline{\overline{T}} \leqslant \mathfrak{m}$, then we put

$$h^*(x_t) = h\big(D_1(x_t)\big)[E_1] \cup \ldots \cup h\big(D_{n-1}(x_t)\big)[E_{n-1}].$$

Hence, by the infinite distributivity of the Post algebra $\boldsymbol{R}_{\mathfrak{m}}/\Delta_{\mathfrak{m}}$ (see [2]),

$$\bigcup_{t\in T} h^*(x_t) = \bigcup_{t\in T} h\big(D_1(x_t)\big)[E_1] \cup \ldots \cup \bigcup_{t\in T} h\big(D_{n-1}(x_t)\big)[E_{n-1}]$$
$$= h\big(\bigcup_{t\in T} D_1(x_t)\big)[E_1] \cup \ldots \cup h\big(\bigcup_{t\in T} D_{n-1}(x_t)\big)[E_{n-1}].$$

Since the representations $(+)$ are monotonic, it follows that

$$\bigcup_{t\in T} D_i(x_t) = D_i\big(\bigcup_{t\in T} x_t\big), \quad i = 1, \ldots, n-1$$

(see [2], p. 133). Consequently

$$\bigcup_{t\in T} h^*(x_t) = h\big(D_1(\bigcup_{t\in T} x_t)\big)[E_1] \cup \ldots \cup h\big(D_{n-1}(\bigcup_{t\in T} x_t)\big)[E_{n-1}] = h^*(\bigcup_{t\in T} x_t).$$

In a similar way one can show that $\bigcap_{t\in T} h^*(x_t) = h^*(\bigcap_{t\in T} x_t)$.

Since, by the hypothesis, h maps B onto $\boldsymbol{F}_{\mathfrak{m}}/\Delta_{\mathfrak{m}}$, it follows that then isomorphism h^* maps P onto $\boldsymbol{R}_{\mathfrak{m}}/\Delta_{\mathfrak{m}}$. This completes the proof.

COROLLARY. *Every σ-complete Post algebra is σ-representable.*

It is a generalization of the Loomis-Sikorski theorem saying that every σ-complete Boolean algebra is σ-representable.

REFERENCES

[1] E. Čech, *On bicompact spaces*, Annals of Mathematics 38 (1937), p. 823-844.

[2] G. Epstein, *The lattice theory of Post algebras*, Transactions of the American Mathematical Society 95 (1960), p. 300-317.

[3] J. L. Kelley, *General topology*, Princeton - New Jersey 1960.

[4] P. C. Rosenbloom, *Post algebras I. Postulates and general theory*, American Journal of Mathematics 64 (1942), p. 167-185.

[5] R. Sikorski, *Boolean algebras*, Berlin-Göttingen-Heidelberg 1960.

[6] T. Traczyk, *Axioms and some properties of Post algebras*, Colloquium Mathematicum 10 (1963), p. 193-209.

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