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ON A PROPERTY OF CONTINUOUS HOMOGENEOUS RANDOM FIELDS

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The subject of this paper are continuous homogeneous random fields in R^2 introduced in the papers of Yaglom [2], Itô [3] and Chiang-Tse-Pei [4]. They are analogous to the complex time dependent wide sense stationary stochastic processes.

Definition 1. The family of complex random variables $x(s, t)$, s and t real numbers, is a *continuous homogeneous random field* if the following conditions hold:

$$E|x(s,t)|^2 < +\infty, \tag{i}$$

$$\lim_{h_1,h_2\to 0} E|x(s+h_1, t+h_2)-x(s,t)|^2 = 0, \tag{ii}$$

and the function

$$E\{x(s+m, t+n)\overline{x(m,n)}\} \tag{iii}$$

does not depend on m and n.

The function given under (iii) shall be denoted by $B_x(s, t)$.

The function $B_x(s, t)$, called the *correlation function* of the continuous homogeneous random field $\{x(s, t)\}$, is continuous, positive definite, and can be represented by

$$B_x(s,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(s\lambda+t\mu)}dF_x(\lambda,\mu),$$

where $F_x(\lambda, \mu)$ is a non-normed two-dimensional distribution function, which, in analogy with time-dependent stochastic processes, shall be called the *spectral function* of the field $\{x(s, t)\}$.

It is well known [2] that every homogeneous random field $\{x(s, t)\}$ has the spectral representation

$$x(s,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(s\lambda+t\mu)}dZ_x(\lambda,\mu), \tag{+}$$

where $Z_x(\lambda, \mu)$ is a random function of two variables with orthogonal increments such that

$$E\{Z_x(S_1)\overline{Z_x(S_2)}\} = \iint_{S_1 \times S_2} dF_x(\lambda, \mu).$$

In our paper we need to express some statements about random variables in the spaces with scalar product. For all such spaces the scalar product is defined by

$$(x, y) = E\{x\bar{y}\},$$

and by convergence we always mean the convergence in mean-square. With such a scalar product the values $x(s, t)$ of random field $\{x(s, t)\}$ form a Hilbert space which we shall denote by $\hat{H}$.

For introducing the notion of filter transformation of random fields which is analogous to such a notion relating to stationary processes [5], it is necessary to introduce some restrictions for random fields. Namely, in paper [4] there have been introduced the notions of singular and regular random fields, which correspond to the notions of deterministic and completely non-deterministic stochastic processes, respectively.

The named notions are basic for us, because we want to formulate in terms of a filter transformation of a homogeneous random field $\{x(s, t)\}$ a property which characterizes such a field.

Let $\{x(s, t)\}$ be a continuous homogeneous random field; by H_x we shall denote (according to [4]) the smallest closed linear space which contains all values $x(m, n)$; by $H_x(t)$ we shall denote the smallest closed linear subspace of H_x which contains all $x(m, n)$, such that $-\infty < m < +\infty$ and $n \leqslant t$.

Let us put $S_x = \bigcap_t H_x(t)$.

Definition 2. A continuous homogeneous random field $\{x(s, t)\}$ is *singular* if $S_x = H_x$.

Using the well known theorem of Rellich [6], which gives the representation for the elements of Hilbert space, we get the unique representation

$$x(s, t) = \eta(s, t) + \xi(s, t)$$

of the element $x(s, t)$ of the random field $\{x(s, t)\}$ such that $\xi(S, t) \in H_x(0)$ and $\eta(s, t)$ is orthogonal to $H_x(0)$.

Let us put

$$r_x^2(s, t) = \|\eta(s, t)\|^2.$$

It is clear that

$$r_x^2(s, t) = r_x^2(s', t) \equiv r_x^2(t),$$

and

$$r_x^2(t_1) \leqslant r_x^2(t_2) \quad \text{for} \quad t_1 \leqslant t_2,$$

whence it follows that the limit

$$\lim_{t \to +\infty} r_x^2(t)$$

exists. Let us denote this limit by σ_∞^2.

Definition 3. Continuous homogeneous random field $\{x(s,t)\}$ is *regular* if

$$\sigma_\infty^2 = E|x(s,t)|^2 = \|x\|^2.$$

Let $h > 0$. By $\mathrm{proj}_{H_x(t-h)} x(s,t)$ we shall denote the projection of $x(s,t)$ on the space $H_x(t-h)$.

Let us put

$$\hat{x}_h(s,t) = x(s,t) - \mathrm{proj}_{H_x(t-h)} x(s,t).$$

Now, given a continuous homogeneous random field $\{x(s,t)\}$ with spectral representation $(+)$, there exists a function

$$C_h(\lambda,\mu) \in L^2\big(dF_x(\lambda,\mu)\big)$$

such that

$$\hat{x}_h(s,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(s\lambda+t\mu)} C_h(\lambda,\mu)\, dZ_x(\lambda,\mu).$$

Moreover, $\{\hat{x}_h(s,t)\}$ is a continuous homogeneous random field with spectral function

$$F_{\hat{x}_h}(\lambda,\mu) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |C_h(\lambda,\mu)|^2 dF_x(\lambda,\mu).$$

Definition 4. We shall call the so defined continuous homogeneous random field $\{\hat{x}_h(s,t)\}$ the *filter transformation* of the continuous homogeneous random field $\{x(s,t)\}$.

This notion is quite analogous to the notion of the filter transformation of the stationary (in wide sense) time-dependent stochastic processes [5].

Using the above notions and definitions we are able to give the following property of regular continuous homogeneous random fileds:

THEOREM. *For every regular continuous homogeneous random field $\{x(s,t)\}$ there exists a regular continuous homogeneous random filed $\{\tilde{x}(s,t)\}$ such that the random field $\{x(s,t)\}$ is a filter transformation of $\{\tilde{x}(s,t)\}$, and we have the representation*

$$x(s,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(s\lambda+t\mu)} C(\lambda,\mu)\, d\xi(\lambda,\mu),$$

where $d\xi(\lambda, \mu)$ is a random Lebesgue's measure, and

$$C(\lambda, \mu) \in L^2\big(dF(\lambda, \mu)\big),$$

F being the spectral function of the field $\{\tilde{x}(s, t)\}$.

Proof. In virtue of a result of [4], we can represent the regular continuous homogeneous random field $\{x(s, t)\}$ in the canonical form

$$(*) \qquad x(s, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\lambda s} g(\lambda, t-\mu)\, d\eta(\lambda, \mu),$$

where

$$g(\lambda, \mu) = \frac{1}{\sqrt{2\pi}} \underset{A\to+\infty}{\text{l. i. m.}} \int_{-A}^{A} e^{i\mu u} C(\lambda, \mu)\, du$$

with

$$(**) \qquad C(\lambda, \mu) = \lim_{v\to 0-} C(\lambda, u+iv),$$

and

$$({}^*{}_*{}^*) \qquad C(\lambda, \omega)$$

$$= k(\lambda) \exp\left\{-\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{1+\mu\omega}{\mu-\omega} \frac{\log\big(dF_x(\lambda, \mu)/dF_x(\lambda, +\infty)d\mu\big)}{1+\mu^2}\, d\mu\right\},$$

$$\operatorname{Im}\{\omega\} < 0,$$

$k(\lambda)$ being a measurable complex function satisfying the condition $|k(\lambda)| = 1$.

Starting from $(*)$ which holds for the considered random field $\{x(s, t)\}$, and using $(**)$ and $({}^*{}_*{}^*)$, we are let do the following:

If we put

$$\xi(A) = \iint_A \frac{dZ_x(\lambda, \mu)}{C(\lambda, \mu)},$$

where Z is the random function of two variables in the spectral representation $(+)$ of the continuous homogeneous random field $\{x(s, t)\}$, then the function $\xi(A)$ has the following properties:

(i) $\xi(A)$ is a random variable as a consequence of the fact that $Z_x(\lambda, \mu)$ is random function;

(ii) if the sets A_1 and A_2 are disjoint, then

$$\xi(A_1+A_2) = \iint_{A_1+A_2} \frac{dZ_x(\lambda, \mu)}{C(\lambda, \mu)} = \iint_{A_1} + \iint_{A_2} = \xi(A_1) + \xi(A_2),$$

as a consequence of the fact that $Z_x(\lambda, \mu)$ is a function with orthogonal increments;

(iii) for arbitrary measurable sets A_1 and A_2 we have

$$E\{\xi(A_1)\overline{\xi(A_2)}\} = \iint_{A_1\times A_2} dF_x(\lambda, +\infty)d\mu.$$

From (i)-(iii) it follows that $\xi(A)$ is a random Lebesgue measure characterizing the random field which we shall denote by $\{\tilde{x}(s,t)\}$.

All these conclusions imply that the regular continuous homogeneous random field $\{x(s,t)\}$ has the representation

$$x(s,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(s\lambda+t\mu)}C(\lambda,\mu)d\xi(\lambda,\mu),$$

where $C(\lambda, \mu)$ satisfies the conditions (**) and ($*_**$) and

$$C(\lambda,\mu)\in L^2\big(d\xi(\lambda,\mu)\big) \subset L^2\big(dF(\lambda,\mu)\big),$$

F being the spectral function of the field $\{\tilde{x}(s,t)\}$ characterized by the random Lebesgue measure $\xi(A)$.

It is easy to see that for the field $\{\tilde{x}(s,t)\}$ we have

$$\sigma_\infty^2 = E|\tilde{x}(s,t)|^2 = \|\tilde{x}\|^2,$$

i. e. $\{\tilde{x}(s,t)\}$ is a regular random field.

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