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ON A THEOREM OF TOEPLITZ

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Let X be a locally bicompact Hausdorff space. By $C(X)$ we denote the set of real functions defined and continuous on X with bicompact carriers and by $C^+(X)$ the set of functions continuous and non-negative on X with bicompact carriers. Let $J(f)$ be a distributive functional defined on $C(X)$ and non-negative on $C^+(X)$. Let μ and $\int f(v)\,d\mu$ denote the Lebesgue measure and the Lebesgue integral generated by the functional J ([4], § 6, [2]). A function $f(x)$ defined on X shall be called *convergent in* ∞ *to the number* ξ if for any $\varepsilon > 0$ the set $\{x: |f(x)-\xi| \geqslant \varepsilon\}$ is contained in a bicompact set.

Definition 1. Let $S = (S_\tau)$ $(0 \leqslant \tau < \infty)$ be a family of bicompact subsets of the space X such that $S_{\tau'} \subset S_{\tau''}$ if $\tau' < \tau''$ and $\bigcup_{0\leqslant\tau<\infty} S_\tau = X$ (1). The improper integral

$$(S)\int f(x)\,d\mu = a$$

is said to *exist* if the integrals $\int_{S_\tau} f(x)\,d\mu$ for $0 \leqslant \tau < \infty$ exist and $\lim_{\tau\to\infty} \int_{S_\tau} f(x)\,d\mu = a$.

Definition 2. Let $\Phi = \big(\Phi(t, x)\big)$ $(0 \leqslant t < T \leqslant \infty)$ denote a family of continuous functions defined on X. A function $f(x)$ defined on X is said to be *limitable by the method* $M = M(SJ; \Phi)$ *to the number* ξ if

1° the integrals $\omega_t = (S)\int \Phi(t, x) f(x)\,d\mu$ exist for every $t \in [0, T)$ and

2° the limit $\lim_{t\to T-} \omega_t = \xi$ exists.

Evidently, by specifying the space X, the classes S and Φ, we can obtain some well-known classes of limitability methods for number sequences or of functions defined on the semi-straight line (see e. g. [3]). The case considered here is, however, much more general, for it includes even the limitability of functions defined in non-metric spaces.

(1) A space for which such a class of sets exists is called a *σ-bicompact space*.

Definition 3. A function defined on X is said to be *locally bounded*, if for every point $x_0 \in X$ there exists a neighbourhood $U(x_0)$ in which this function is bounded.

THEOREM (of Toeplitz). *In order that a method $M = M(JS; \Phi)$ limitates all μ-measurable, convergent in ∞ and locally bounded functions to their ordinary limits, it is necessary and sufficient that the following conditions are satisfied:*

1° $\lim\limits_{t\to T-} (S)\int \Phi(t, x)\, d\mu = 1$;

2° $\lim\limits_{t\to T-} \int\limits_{A} \Phi(t, x)\, d\mu = 0$ *for every measurable set A contained in some bicompact set*;

3° $\overline{\lim\limits_{t\to T-}}\, (S)\int |\Phi(t, x)|\, d\mu < \infty$.

Proof of necessity. The necessity of conditions 1° and 2° is evident. To prove the necessity of condition 3° we complete the space X with the point x_∞ into a bicompact space to be denoted by X_∞. The set of functions continuous on X and having a limit in ∞ may be treated as a set of functions continuous on X_∞. We denote it by $C(X_\infty)$. The set $C(X_\infty)$ is a Banach space with the norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)|.$$

The functional

$$A_{t,\tau}(f) = \int\limits_{S_\tau} \Phi(t, x) f(x)\, d\mu$$

is linear in $C(X_\infty)$. We shall show that its norm is equal to $\int\limits_{S_\tau} |\Phi(t, x)|\, d\mu$.

Since the set S_τ is bicompact, we have $\mu(S_\tau) = K < \infty$ and there exists, for every $\eta > 0$, an open set $U \supset S_\tau$ such that $\mu(U - S_\tau) < \eta$.

We denote by A_1 and A_2 the subsets of the set S_τ in which $\Phi(t, x) \geqslant \varepsilon/K$ or $\Phi(t, x) \leqslant -\varepsilon/K$, respectively.

In view of the continuity of the function Φ, the sets A_i are bicompact. Thus there exists a function $\varphi(x)$ continuous on X and such that $|\varphi(x)| \leqslant 1$ and

$$\varphi(x) = \begin{cases} 1 & \text{for} \quad x \in A_1, \\ -1 & \text{for} \quad x \in A_2, \\ 0 & \text{for} \quad x \notin U. \end{cases}$$

Obviously $\varphi \in C(X_\infty)$. Moreover, we have

$$A_{t,\tau}(\varphi) \geqslant \int\limits_{S_\tau} |\Phi(t, x)|\, d\mu - \varepsilon.$$

Since $\|\varphi\|_\infty = 1$, this implies

$$\|A_{t,\tau}\| = \int_{S_\tau} |\Phi(t,x)|\,d\mu.$$

The functional

$$A_t(f) = \lim_{\tau\to\infty} A_{t,\tau}(f) = (S)\int \Phi(t,x)f(x)\,d\mu$$

is defined in $C(X_\infty)$, therefore according to Banach-Steinhaus theorem ([1], Th. 5, p. 80) we have

$$\sup_{0\leqslant\tau<\infty} \|A_{t,\tau}\| = (S)\int |\Phi(t,x)|\,d\mu < \infty.$$

Let us now fix a set S_τ and an open set $U \supset S_\tau$ such that

$$\int_{U-S_\tau} |\Phi(t,x)|\,d\mu < \varepsilon.$$

We have

$$A_t(\varphi) = A_{t,\tau}(\varphi) + \int_{U-S_\tau} \Phi(t,x)\varphi(x)\,d\mu \geqslant \|A_{t,\tau}\| - \varepsilon + \int_{U-S_\tau} \Phi(t,x)\varphi(x)\,d\mu.$$

But

$$\int_{U-S_\tau} \Phi(t,x)\varphi(x)\,d\mu > -\int_{U-S_\tau} |\Phi(t,x)|\,d\mu > -\varepsilon.$$

Therefore

$$A_t(\varphi) \geqslant \|A_{t,\tau}\| - 2\varepsilon.$$

Letting ε tend to 0 and τ tend to T_0, we obtain immediately

$$\|A_t\| = (S)\int |\Phi|(t,x)\,d\mu \quad \text{for} \quad t\in[0,T).$$

Since by assumption the limit $\lim_{t\to T-} A_t(f)$ exists for every $f\in C(X_\infty)$, by Banach-Steinhaus theorem we have

$$\overline{\lim_{t\to T-}}\, \|A_t\| < \infty,$$

which ends the proof of the necessity of condition 3°.

Proof of sufficiency. We show at first that for every locally bounded and μ-measurable function and for every bicompact set A

$$\lim_{t\to T-} \int_A \Phi(t,x)f(x)\,d\mu = 0$$

holds.

Let a $\delta > 0$ be given. Since $f(x)$ is locally bounded, it is bounded on the set A. On this set we have, say,

$$m \leqslant f(x) \leqslant M.$$

We fix $a_1, a_2, \ldots, a_N$ such that

$$m = a_1 < a_2 < a_3 \ldots < a_N = M \quad \text{and} \quad a_{i+1} - a_i < \delta/2K,$$

where

$$K = \sup_{0 \leqslant t < T} (S) \int |\Phi(t, x)| d\mu \ (^2).$$

We set

$$A_i = \{x \in A : a_i \leqslant f(x) < a_{i+1}\} \cap A, \quad i = 1, 2, \ldots, N-2,$$
$$A_{N-1} = \{x \in A : a_{N-1} \leqslant f(x) \leqslant M\} \cap A$$

and

$$\psi(x) = \begin{cases} a_i & \text{for} \quad x \in A_i, \\ 0 & \text{for} \quad x \notin A. \end{cases}$$

Obviously the function $\psi(x)$ is measurable and we have $f(x) - \psi(x) < \delta/2K$ on the set A. Moreover, by 2° we have

$$\Big|\int_A \Phi(t, x) f(x) d\mu\Big| = \Big|\int_A \Phi(t, x)[f(x) - \psi(x)] d\mu + \sum_{i=1}^{N-1} a_i \int_{A_i} \Phi(t, x) d\mu\Big|$$
$$\leqslant \frac{\delta}{2K} \int_A |\Phi(t, x)| d\mu + \frac{\delta}{2} < \delta$$

for t sufficiently large.

Now, let $f(x)$ be a locally bounded and μ-measurable function and let $\lim_{x \to \infty} f(x) = \alpha$. Let an $\varepsilon > 0$ be given. Then there exists a bicompact set Z such that

$$\{x : |f(x) - \alpha| \geqslant \varepsilon/K\} \subset Z.$$

We denote by CZ the complement of Z to the space X and by $\eta_A(x)$ the characteristic function of A. Then we have

$$(S)\int \Phi(t, x)[f(x) - \alpha] d\mu$$
$$= \int_Z \Phi(t, x)[f(x) - \alpha] + (S)\int \Phi(t, x)[f(x) - \alpha] \ \eta_{CZ}(x) d\mu.$$

(2) Without diminishing the generality of our argument we may assume that $\sup_{0 \leqslant t < T} (S)\int |\Phi(t, x)| d\mu < \infty$.

According to what has been said above the first integral tends to zero as $t \to T-$. From the estimation

$$\left|(S)\int \Phi(t, x)[f(x)-a]\eta_{(z}(x)\,d\mu\right| \leqslant (S)\int |\Phi(t, x)|\,d\mu\cdot\frac{\varepsilon}{K}$$

of the second integral we conclude that its absolute value is smaller than ε. By additivity of the functional

$$M(f) = \lim_{t\to\infty}(S)\int \Phi(t, x)f(x)\,d\mu$$

and condition 1° it follows that the function $f(x)$ is limitable in ∞ to the number a, which ends the proof of the theorem.

REFERENCES

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[2] N. Bourbaki, *Intégration, Éléments de Mathématique VII*, Paris 1952.

[3] G. H. Hardy, *Divergent series*, Oxford 1956.

[4] M. A. Наймарк, *Нормированные кольца*, Москва 1956.

[5] O. Toeplitz, *Über allgemeine lineare Mittelbildungen*, Prace Matematyczno-Fizyczne 22 (1911), p. 113-119.

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