ON A LEMMA OF S. KACZMARZ

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In the present note we shall deal with transferring a Kaczmarz Lemma (1), concerning some relations between the \( n \)-th arithmetic means of numerical sequences, to the case of logarithmic means. When studying this case, it appeared necessary to replace the second assumption of Kaczmarz lemma by a more restrictive condition.

It seems interesting to find out whether the lemma presented here holds (as far as it is possible in the general case) also without any additional restriction (P 471). It would be also of interest to generalize this lemma by proving it e.g. in the case of weighted means or in the case of the Nörlund method of summability (P 472).

The lemma which follows may find an application in the investigation of the summability of orthogonal series by the method of logarithmic means, with regard to the order of magnitude of Lebesgue functions.

**Lemma.** Let there be given an increasing sequence of positive numbers \( \{\lambda_n\} \) and a series \( \sum_{n=1}^{\infty} a_n \).

Denote by \( \tau_n, T_n \) and \( T_n^{(1)} \) the first logarithmic means of the series

\[
\sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n} \quad \text{and} \quad \sum_{n=1}^{\infty} S_n^{(1)} \lambda_n \frac{1}{\lambda_n},
\]

respectively. Here \( S_n^{(1)} = s_1 + s_2 + \ldots + s_n \), where \( s_n = a_1 + a_2 + \ldots + a_n \).

We assume the following two conditions to be satisfied:

1. \( \tau_n = O(\lambda_n) \) for \( n \to \infty \).
2. \( A1/\lambda_n = O(1/n \lambda_n \log n) \).

Then we have

\[
T_n = T_n^{(1)} + O(1) \quad \text{for} \quad n \to \infty.
\]

Proof. The definition of $T_n$ implies

$$ T_n = \frac{1}{\log n} \sum_{k=1}^{n} a_k \sum_{v=k}^{n} \frac{1}{v}. $$

Hence

$$ T_n = \frac{1}{\lambda_n \log n} \sum_{k=1}^{n} a_k \sum_{v=k}^{n} \frac{1}{v} + \frac{1}{\log n} \sum_{k=1}^{n} a_k \left( \frac{1}{\lambda_k} - \frac{1}{\lambda_n} \right) \sum_{v=k}^{n} \frac{1}{v}, $$

and introducing the symbol $\tau_n$ we obtain

(1) $$ T_n = \frac{\tau_n}{\lambda_n} + \frac{1}{\log n} \sum_{k=1}^{n} a_k \left( \frac{1}{\lambda_k} - \frac{1}{\lambda_n} \right) \sum_{v=k}^{n} \frac{1}{v}. $$

Let us write

(2) $$ \eta_k = \begin{cases} \sum_{v=k}^{n} \frac{1}{v} \left( \frac{1}{\lambda_k} - \frac{1}{\lambda_n} \right) & \text{for } k = 1, 2, \ldots, n, \\ 0 & \text{for } k > n. \end{cases} $$

Then for $k \leq n$

(3) $$ \Delta \eta_k = \frac{1}{k \lambda_k} + \sum_{v=k+1}^{n} \frac{1}{v} \frac{\Delta}{\lambda_k} - \frac{1}{k \lambda_n}, $$

and

$$ \Delta^2 \eta_k = \Delta \eta_k - \Delta \eta_{k+1} = \frac{1}{k \lambda_k} - \frac{1}{(k+1) \lambda_{k+1}} - $$

$$ \left( \Delta \frac{1}{\lambda_{k+1}} \right) \sum_{v=k+2}^{n} \frac{1}{v} - \frac{1}{k \lambda_n} + \frac{1}{(k+1) \lambda_n} + \left( \Delta \frac{1}{\lambda_k} \right) \sum_{v=k+1}^{n} \frac{1}{v}. $$

Hence

(4) $$ \Delta^2 \eta_k = -\frac{1}{\lambda_k} \Delta \frac{1}{k} + \frac{1}{k+1} \Delta \frac{1}{\lambda_k} + \Delta \frac{1}{k \lambda_k} + \left( \Delta^2 \frac{1}{\lambda_k} \right) \sum_{v=k+2}^{n} \frac{1}{v}. $$

Applying equalities (2), (3) and $\eta_n = \Delta \eta_n = \Delta^2 \eta_n = 0$ we can write the second term on the right-hand side of formula (1) in the form

$$ \frac{1}{\log n} \sum_{k=1}^{n} a_k \eta_k = \frac{\eta_n}{\log n} \sum_{k=1}^{n} a_k + \frac{1}{\log n} \sum_{k=1}^{n-1} s_k \Delta \eta_k $$

$$ = \frac{1}{\log n} \sum_{k=1}^{n} s_k \Delta \eta_k = \Delta \eta_n \theta_n^{(1)} + \frac{1}{\log n} \sum_{k=1}^{n-1} \theta_k^{(1)} \Delta^2 \eta_k $$

$$ = \frac{1}{\log n} \sum_{k=1}^{n} \theta_k^{(1)} \Delta^2 \eta_k. $$
Thus
\[
\frac{1}{\log n} \sum_{k=1}^{n} a_k \eta_k = \frac{1}{\log n} \sum_{k=1}^{n} S_k^{(1)} \Delta^2 \eta_k.
\]

If we put here the value given in formula (4) in place of \( \Delta^2 \eta_k \), we obtain
\[
\frac{1}{\log n} \sum_{k=1}^{n} a_k \eta_k = -\frac{1}{\lambda_n \log n} \sum_{k=1}^{n} S_k^{(1)} \Delta \frac{1}{k} + \frac{1}{\log n} \sum_{k=1}^{n} S_k^{(1)} \frac{1}{k+1} \Delta \frac{1}{\lambda_k} + \frac{1}{\log n} \sum_{k=1}^{n} S_k^{(1)} \left( \frac{1}{\lambda_k} \right)^2 + \frac{1}{\log n} \sum_{k=1}^{n} S_k^{(1)} \left( \frac{1}{\lambda_k} \right)^{1+v\Delta^2} \sum_{v=k+2}^{n} \frac{1}{v} = S_1 + S_2 + S_3 + S_4.
\]

Here \( S_1, S_2, S_3 \) and \( S_4 \) stand for the first, second, third and fourth terms in the last sum, respectively. Now we proceed to the estimation of \( S_1, S_2, S_3 \) and \( S_4 \):
\[
S_1 = -\frac{1}{\lambda_n \log n} \sum_{k=1}^{n} s_k \sum_{v=k}^{n} \left( \frac{1}{v} - \frac{1}{v+1} \right) = -\frac{1}{\lambda_n \log n} \sum_{k=1}^{n} s_k \left( \frac{1}{k} - \frac{1}{n+1} \right)
\]
\[
= -\frac{1}{\lambda_n \log n} \sum_{k=1}^{n} \frac{s_k}{k} + \frac{1}{\lambda_n \log n} \sum_{k=1}^{n} s_k
\]
\[
= -\frac{\tau_n}{\lambda_n} + \frac{1}{(n+1) \lambda_n \log n} \sum_{k=1}^{n} s_k \cdot k
\]
\[
= -\frac{\tau_n}{\lambda_n} + \frac{n \tau_n}{(n+1) \lambda_n} - \frac{1}{(n+1) \lambda_n \log n} \sum_{k=1}^{n} \sum_{i=1}^{k} s_i
\]
\[
= -\frac{\tau_n}{\lambda_n} + \frac{n \tau_n}{(n+1) \lambda_n} + \frac{\tau_n}{(n+1) \lambda_n} - \frac{1}{(n+1) \lambda_n \log n} \sum_{k=1}^{n} \sum_{i=1}^{k} s_i.
\]

Hence the formula
\[
\frac{1}{(n+1) \lambda_n \log n} \sum_{k=1}^{n} \sum_{i=1}^{k} s_i = O(1) + \frac{1}{(n+1) \lambda_n \log n} \sum_{k=3}^{n} \tau_k \log k
\]
and assumption 1 imply
\[
S_2 = O(1) \quad \text{for} \quad n \to \infty,
\]
\[
S_2 = \frac{1}{\log n} \sum_{k=1}^{n} \frac{S_k^{(1)}}{k+1} \Delta \frac{1}{\lambda_k} = \frac{1}{\log n} \sum_{k=1}^{n} \frac{s_k}{k} \sum_{i=k}^{n} \frac{1}{i+1} \Delta \frac{1}{\lambda_i}
\]
\[
= \frac{n \tau_n}{n+1} + \frac{1}{\log n} \sum_{k=1}^{n} \left[ \Delta \left( k \sum_{i=k}^{n} \frac{1}{i+1} \Delta \frac{1}{\lambda_i} \right) \right] \sum_{v=1}^{k} s_v.
\]
Evidently we have
\[
\Delta \left( k \sum_{i=k}^{n} \frac{1}{i+1} \Delta \frac{1}{\lambda_i} \right) = \frac{k \Delta}{\lambda_k} \frac{1}{k+1} - \sum_{i=k+1}^{n} \frac{1}{i+1} \Delta \frac{1}{\lambda_i}.
\]
Then from assumption 2 it follows
\[
\Delta \left( k \sum_{i=k}^{n} \frac{1}{i+1} \Delta \frac{1}{\lambda_i} \right) = O \left( \frac{1}{k \lambda_k \log k} \right) + \sum_{i=k+1}^{\infty} O \left( \frac{1}{i^2 \lambda_i \log i} \right)
\]
\[
= O \left( \frac{1}{k \lambda_k \log k} \right) + O \left( \frac{1}{k \lambda_k \log k} \right) = O \left( \frac{1}{k \lambda_k \log k} \right).
\]
This estimation together with assumptions 1 and 2 implies
\[
|S_3| = O(\lambda_n) \cdot O \left( \frac{1}{n \lambda_n \log n} \right) + \frac{1}{\log n} \sum_{k=3}^{n} \log k \cdot O(\lambda_k) \cdot O \left( \frac{1}{k \lambda_k \log k} \right) + O(1)
\]
\[
= O(1) + \frac{1}{\log n} \sum_{k=3}^{n} O \left( \frac{1}{k} \right).
\]
Hence we conclude that
\[
S_3 = O(1) \quad \text{for} \quad n \to \infty,
\]
\[
S_3 = \frac{1}{\log n} \sum_{k=1}^{n} \Delta \left( \frac{1}{k \lambda_k} \right) = \frac{1}{\log n} \sum_{k=1}^{n} s_k \sum_{v=k}^{n} \Delta \left( \frac{1}{v \lambda_v} \right)
\]
\[
= \frac{1}{\log n} \sum_{v=1}^{n} \frac{s_k}{k} \sum_{v=k}^{n} \Delta \left( \frac{1}{v \lambda_v} \right)
\]
\[
= n \tau_n \Delta \left( \frac{1}{n \lambda_n} \right) + \frac{1}{\log n} \sum_{k=1}^{n} \Delta \left[ k \sum_{v=k}^{n} \Delta \left( \frac{1}{v \lambda_v} \right) \right] \sum_{v=1}^{n} s_v.
\]
Since
\[
\Delta \left( \frac{1}{v \lambda_v} \right) = \frac{1}{v \lambda_v} - (v+1) \lambda_{v+1} \lambda_v \Delta \left( \frac{1}{v \lambda_v} \right)
\]
\[
= k \left[ \sum_{v=k}^{n} \Delta \left( \frac{1}{v \lambda_v} \right) - \sum_{v=k+1}^{n} \Delta \left( \frac{1}{v \lambda_v} \right) \right] - \sum_{v=k+1}^{n} \left[ \frac{1}{v \lambda_v} - (v+1) \lambda_{v+1} \lambda_v \right]
\]
\[
= k \Delta \left( \frac{1}{k \lambda_k} \right) - \sum_{v=k+1}^{n} \left[ \frac{1}{v \lambda_v} - \frac{1}{(v+1) \lambda_{v+1}} \right]
\]
\[
= \frac{1}{\lambda_k} - \frac{k}{(k+1) \lambda_{k+1}} - \frac{1}{(k+1) \lambda_{k+1}} + \frac{1}{(n+1) \lambda_{n+1}} = \Delta \frac{1}{\lambda_k} + \frac{1}{(n+1) \lambda_{n+1}}
\]
and
\[ n \tau_n A \left( \frac{1}{n \lambda_n} \right) = \frac{\tau_n}{\lambda_n} - \frac{n \tau_n}{(n+1) \lambda_{n+1}} - \frac{\tau_n}{(n+1) \lambda_{n+1}} + \frac{\tau_n}{(n+1) \lambda_{n+1}} \]
\[ = \tau_n A \frac{1}{\lambda_n} + \frac{\tau_n}{(n+1) \lambda_{n+1}}, \]
we have
\[ S_3 = \tau_n A \frac{1}{\lambda_n} + \frac{\tau_n}{(n+1) \lambda_{n+1}} + \frac{1}{\log n} \sum_{k=3}^{n-1} \tau_k \log k A \frac{1}{\lambda_k} + \]
\[ + \frac{1}{(n+1) \lambda_{n+1} \log n} \sum_{k=3}^{n-1} \tau_k \log k + O(1). \]
Thus assumptions 1 and 2 imply that
\[ S_3 = O \left( \frac{\tau_n}{\lambda_n} \right) + O \left( \frac{\tau_n}{\lambda_{n+1}} \right) + \frac{1}{\log n} \sum_{k=3}^{n-1} \tau_k \log k O \left( \frac{1}{k \lambda_k \log k} \right) + \]
\[ + \frac{1}{n+1} \sum_{k=3}^{n-1} O(1) + O(1) = O(1), \]
i. e.,
\[ S_3 = O(1) \quad \text{for} \quad n \to \infty. \]

We have
\[ S_4 = \frac{1}{\log n} \sum_{k=1}^{n} S_k^{(1)} A^2 \left( \frac{1}{\lambda_k} \right) \frac{1}{k} = \frac{1}{\log n} \sum_{k=1}^{n} S_k^{(1)} A^2 \left[ \frac{1}{\lambda_k} \right] \left( \frac{1}{k} + \frac{1}{k+1} \right) \]
\[ = T_n^{(1)} - U_n, \]
where \( T_n^{(1)} \) and \( U_n \) denote the first and the second expression on the right-hand side of the last equality. Thus we have
\[ S_4 = T_n^{(1)} - U_n. \]

Since
\[ A \left\{ k \sum_{i=k}^{n} \left( \frac{1}{i} + \frac{1}{i+1} \right) A^2 \frac{1}{\lambda_i} \right\} \]
\[ = A^2 \frac{1}{\lambda_k} + \frac{k}{k+1} A^2 \frac{1}{\lambda_k} - \sum_{i=k+1}^{n} \left( \frac{1}{i} + \frac{1}{i+1} \right) A^2 \frac{1}{\lambda_i} \]
and assumption 2 implies
\[ \Delta \left\{ k \sum_{i=k}^{n} \left( \frac{1}{i} + \frac{1}{i+1} \right) \Delta^2 \frac{1}{\lambda_i} \right\} = O \left( \frac{1}{k \lambda_k \log k} \right) + \sum_{i=k+1}^{\infty} O \left( \frac{1}{i^2 \lambda_i \log i} \right) = O \left( \frac{1}{k \lambda_k \log k} \right) + O \left( \frac{1}{k \lambda_k \log k} \right) = O \left( \frac{1}{k \lambda_k \log k} \right), \]
we have for
\[
U_n = \frac{1}{\log n} \sum_{i=k}^{n} \frac{s_k}{k} \sum_{i=k}^{n} \Delta^2 \left[ \frac{1}{\lambda_i} \right] \left( \frac{1}{i} + \frac{1}{i+1} \right) \\
= \frac{1}{\log n} \Delta^2 \left[ \frac{1}{\lambda_n} \right] \left( \frac{1}{n} + \frac{1}{n+1} \right) \cdot n \sum_{k=1}^{n} \frac{s_k}{k} + \\
+ \frac{1}{\log n} \sum_{k=1}^{n-1} \Delta \left\{ k \sum_{i=k}^{n} \left( \frac{1}{i} + \frac{1}{i+1} \right) \Delta^2 \frac{1}{\lambda_i} \right\} \sum_{v=1}^{k} \frac{s_v}{v},
\]
the following estimation:
\[
U_n = O \left( \frac{r_n}{\lambda_n} \right) + \frac{1}{\log n} \sum_{k=3}^{n-1} O(\lambda_k \log k) O \left( \frac{1}{k \lambda_k \log k} \right) + O(1) = O(1).
\]

From assumption 1, formulas (6), (7), (8), (9), (1) and the relation \( U_n = O(1) \) it follows that
\[
T_n = T_n^{(1)} + O(1) \quad \text{for} \quad n \to \infty,
\]
and this is the required result.