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ON A LEMMA OF S. KACZMARZ

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In the present note we shall deal with transferring a Kaczmarz Lemma (1), concerning some relations between the n-th arithmetic means of numerical sequences, to the case of logarithmic means. When studying this case, it appeared necessary to replace the second assumption of Kaczmarz lemma by a more restrictive condition.

It seems interesting to find out whether the lemma presented here holds (as far as it is possible in the general case) also without any additional restriction (**P 471**). It would be also of interest to generalize this lemma by proving it e. g. in the case of weighted means or in the case of the Nörlund method of summability (**P 472**).

The lemma which follows may find an application in the investigation of the summability of orthogonal series by the method of logarithmic means, with regard to the order of magnitude of Lebesgue functions.

LEMMA. *Let there be given an increasing sequence of positive numbers* $\{\lambda_n\}$ *and a series* $\sum_{n=1}^{\infty} a_n$.

Denote by τ_n, T_n *and* $T_n^{(1)}$ *the first logarithmic means of the series*

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n} \quad \text{and} \quad \sum_{n=1}^{\infty} S_n^{(1)} \Delta^2 \frac{1}{\lambda_n},$$

respectively. Here $S_n^{(1)} = s_1+s_2+\ldots+s_n$, *where* $s_n = a_1+a_2+\ldots+a_n$. *We assume the following two conditions to be satisfied:*

1. $\tau_n = O(\lambda_n)$ *for* $n \to \infty$.
2. $\Delta 1/\lambda_n = O(1/n\lambda_n \log n)$.

Then we have

$$T_n = T_n^{(1)} + O(1) \quad \text{for} \quad n \to \infty.$$

(1) See S. Kaczmarz, *Sur la convergence et sommabilité des développements orthogonaux*, Studia Mathematica 1 (1929), p. 87-121; Lemma 5, p. 111.

Proof. The definition of T_n implies

$$T_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{a_k}{\lambda_k} \sum_{\nu=k}^{n} \frac{1}{\nu}.$$

Hence

$$T_n = \frac{1}{\lambda_n \log n} \sum_{k=1}^{n} a_k \sum_{\nu=k}^{n} \frac{1}{\nu} + \frac{1}{\log n} \sum_{k=1}^{n} a_k \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_n}\right) \sum_{\nu=k}^{n} \frac{1}{\nu}$$

and introducing the symbol τ_n we obtain

$$T_n = \frac{\tau_n}{\lambda_n} + \frac{1}{\log n} \sum_{k=1}^{n} a_k \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_n}\right) \sum_{\nu=k}^{n} \frac{1}{\nu}. \tag{1}$$

Let us write

$$\eta_k = \begin{cases} \displaystyle\sum_{\nu=k}^{n} \frac{1}{\nu} \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_n}\right) & \text{for} \quad k = 1, 2, \ldots, n, \\ 0 & \text{for} \quad k > n. \end{cases} \tag{2}$$

Then for $k \leqslant n$

$$\Delta \eta_k = \frac{1}{k\lambda_k} + \sum_{\nu=k+1}^{n} \frac{1}{\nu} \Delta \frac{1}{\lambda_k} - \frac{1}{k\lambda_n} \tag{3}$$

and

$$\Delta^2 \eta_k = \Delta \eta_k - \Delta \eta_{k+1} = \frac{1}{k\lambda_k} - \frac{1}{(k+1)\lambda_{k+1}} -$$

$$- \left(\Delta \frac{1}{\lambda_{k+1}}\right) \sum_{\nu=k+2}^{n} \frac{1}{\nu} - \frac{1}{k\lambda_n} + \frac{1}{(k+1)\lambda_n} + \left(\Delta \frac{1}{\lambda_k}\right) \sum_{\nu=k+1}^{n} \frac{1}{\nu}.$$

Hence

$$\Delta^2 \eta_k = -\frac{1}{\lambda_n} \Delta \frac{1}{k} + \frac{1}{k+1} \Delta \frac{1}{\lambda_k} + \Delta \frac{1}{k\lambda_k} + \left(\Delta^2 \frac{1}{\lambda_k}\right) \sum_{\nu=k+2}^{n} \frac{1}{\nu}. \tag{4}$$

Applying equalities (2), (3) and $\eta_n = \Delta \eta_n = \Delta^2 \eta_n = 0$ we can write the second term on the right-hand side of formula (1) in the form

$$\frac{1}{\log n} \sum_{k=1}^{n} a_k \eta_k = \frac{\eta_n}{\log n} \sum_{k=1}^{n} a_k + \frac{1}{\log n} \sum_{k=1}^{n-1} s_k \Delta \eta_k$$

$$= \frac{1}{\log n} \sum_{k=1}^{n} s_k \Delta \eta_k = \frac{\Delta \eta_n}{\log n} S_n^{(1)} + \frac{1}{\log n} \sum_{k=1}^{n-1} S_k^{(1)} \Delta^2 \eta_k$$

$$= \frac{1}{\log n} \sum_{k=1}^{n} S_k^{(1)} \Delta^2 \eta_k.$$

Thus

$$\frac{1}{\log n}\sum_{k=1}^{n} a_k\eta_k = \frac{1}{\log n}\sum_{k=1}^{n} S_k^{(1)}\Delta^2\eta_k.$$

If we put here the value given in formula (4) in place of $\Delta^2\eta_k$, we obtain

$$\text{(5)}\quad \frac{1}{\log n}\sum_{k=1}^{n} a_k\eta_k = -\frac{1}{\lambda_n\log n}\sum_{k=1}^{n} S_k^{(1)}\Delta\frac{1}{k}+\frac{1}{\log n}\sum_{k=1}^{n}\frac{S_k^{(1)}}{k+1}\Delta\frac{1}{\lambda_k}+$$
$$+\frac{1}{\log n}\sum_{k=1}^{n} S_k^{(1)}\left(\Delta\frac{1}{k\lambda_k}\right)+\frac{1}{\log n}\sum_{k=1}^{n} S_k^{(1)}\left(\Delta^2\frac{1}{\lambda_k}\right)\sum_{\nu=k+2}^{n}\frac{1}{\nu} = S_1+S_2+S_3+S_4.$$

Here S_1, S_2, S_3 and S_4 stand for the first, second, third and fourth terms in the last sum, respectively. Now we proceed to the estimation of S_1, S_2, S_3 and S_4:

$$S_1 = -\frac{1}{\lambda_n\log n}\sum_{k=1}^{n} s_k\sum_{\nu=k}^{n}\left(\frac{1}{\nu}-\frac{1}{\nu+1}\right) = -\frac{1}{\lambda_n\log n}\sum_{k=1}^{n} s_k\left(\frac{1}{k}-\frac{1}{n+1}\right)$$
$$= -\frac{1}{\lambda_n\log n}\sum_{k=1}^{n}\frac{s_k}{k}+\frac{1}{\lambda_n(n+1)\log n}\sum_{k=1}^{n} s_k$$
$$= -\frac{\tau_n}{\lambda_n}+\frac{1}{(n+1)\lambda_n\log n}\sum_{k=1}^{n}\frac{s_k}{k}\cdot k$$
$$= -\frac{\tau_n}{\lambda_n}+\frac{n\tau_n}{(n+1)\lambda_n}-\frac{1}{(n+1)\lambda_n\log n}\sum_{k=1}^{n-1}\sum_{i=1}^{k}\frac{s_i}{i}$$
$$= -\frac{\tau_n}{\lambda_n}+\frac{n\tau_n}{(n+1)\lambda_n}+\frac{\tau_n}{(n+1)\lambda_n}-\frac{1}{(n+1)\lambda_n\log n}\sum_{k=1}^{n}\sum_{i=1}^{k}\frac{s_i}{i}.$$

Hence the formula

$$\frac{1}{(n+1)\lambda_n\log n}\sum_{k=1}^{n}\sum_{i=1}^{k}\frac{s_i}{i} = O(1)+\frac{1}{(n+1)\lambda_n\log n}\sum_{k=3}^{n}\tau_k\log k$$

and assumption 1 imply

$$\text{(6)}\qquad S_1 = O(1)\quad\text{for}\quad n\to\infty,$$

$$S_2 = \frac{1}{\log n}\sum_{k=1}^{n}\frac{S_k^{(1)}}{k+1}\Delta\frac{1}{\lambda_k} = \frac{1}{\log n}\sum_{k=1}^{n}\frac{s_k}{k}k\sum_{i=k}^{n}\frac{1}{i+1}\Delta\frac{1}{\lambda_i}$$
$$= \frac{n\tau_n\Delta\frac{1}{\lambda_n}}{n+1}+\frac{1}{\log n}\sum_{k=1}^{n-1}\left[\Delta\left(k\sum_{i=k}^{n}\frac{1}{i+1}\Delta\frac{1}{\lambda_i}\right)\right]\sum_{\nu=1}^{k}\frac{s_\nu}{\nu}.$$

Evidently we have

$$\Delta\left(k\sum_{i=k}^{n}\frac{1}{i+1}\,\Delta\frac{1}{\lambda_i}\right)=\frac{k\Delta\dfrac{1}{\lambda_k}}{k+1}-\sum_{i=k+1}^{n}\frac{1}{i+1}\,\Delta\frac{1}{\lambda_i}.$$

Then from assumption 2 it follows

$$\Delta\left(k\sum_{i=k}^{n}\frac{1}{i+1}\,\Delta\frac{1}{\lambda_i}\right)=O\left(\frac{1}{k\lambda_k\log k}\right)+\sum_{i=k+1}^{\infty}O\left(\frac{1}{i^2\lambda_i\log i}\right)$$

$$=O\left(\frac{1}{k\lambda_k\log k}\right)+O\left(\frac{1}{k\lambda_k\log k}\right)=O\left(\frac{1}{k\lambda_k\log k}\right).$$

This estimation together with assumptions 1 and 2 implies

$$|S_2|=O(\lambda_n)\cdot O\left(\frac{1}{n\lambda_n\log n}\right)+\frac{1}{\log n}\sum_{k=3}^{n}\log k\cdot O(\lambda_k)\cdot O\left(\frac{1}{k\lambda_k\log k}\right)+O(1)$$

$$=O(1)+\frac{1}{\log n}\sum_{k=3}^{n}O\left(\frac{1}{k}\right).$$

Hence we conclude that

$$S_2=O(1)\quad\text{for}\quad n\to\infty, \tag{7}$$

$$S_3=\frac{1}{\log n}\sum_{k=1}^{n}S_k^{(1)}\Delta\left(\frac{1}{k\lambda_k}\right)=\frac{1}{\log n}\sum_{k=1}^{n}s_k\sum_{\nu=k}^{n}\Delta\left(\frac{1}{\nu\lambda_\nu}\right)$$

$$=\frac{1}{\log n}\sum_{\nu=1}^{n}\frac{s_k}{k}\,k\sum_{\nu=k}^{n}\Delta\left(\frac{1}{\nu\lambda_\nu}\right)$$

$$=n\tau_n\Delta\left(\frac{1}{n\lambda_n}\right)+\frac{1}{\log n}\sum_{k=1}^{n-1}\Delta\left[k\sum_{\nu=k}^{n}\Delta\frac{1}{\nu\lambda_\nu}\right]\sum_{\nu=1}^{k}\frac{s_\nu}{\nu}.$$

Since

$$\Delta\left[k\sum_{\nu=k}^{n}\Delta\left(\frac{1}{\nu\lambda_\nu}\right)\right]=k\sum_{\nu=k}^{n}\Delta\left(\frac{1}{\nu\lambda_\nu}\right)-(k+1)\sum_{\nu=k+1}^{n}\Delta\left(\frac{1}{\nu\lambda_\nu}\right)$$

$$=k\left[\sum_{\nu=k}^{n}\Delta\left(\frac{1}{\nu\lambda_\nu}\right)-\sum_{\nu=k+1}^{n}\Delta\left(\frac{1}{\nu\lambda_\nu}\right)\right]-\sum_{\nu=k+1}^{n}\left[\frac{1}{\nu\lambda_\nu}-\frac{1}{(\nu+1)\lambda_{\nu+1}}\right]$$

$$=k\Delta\left(\frac{1}{k\lambda_k}\right)-\sum_{\nu=k+1}^{n}\left[\frac{1}{\nu\lambda_\nu}-\frac{1}{(\nu+1)\lambda_{\nu+1}}\right]$$

$$=\frac{1}{\lambda_k}-\frac{k}{(k+1)\lambda_{k+1}}-\frac{1}{(k+1)\lambda_{k+1}}+\frac{1}{(n+1)\lambda_{n+1}}=\Delta\frac{1}{\lambda_k}+\frac{1}{(n+1)\lambda_{n+1}}$$

and

$$n\tau_n\Delta\left(\frac{1}{n\lambda_n}\right)=\frac{\tau_n}{\lambda_n}-\frac{n\tau_n}{(n+1)\lambda_{n+1}}-\frac{\tau_n}{(n+1)\lambda_{n+1}}+\frac{\tau_n}{(n+1)\lambda_{n+1}}$$
$$=\tau_n\Delta\frac{1}{\lambda_n}+\frac{\tau_n}{(n+1)\lambda_{n+1}},$$

we have

$$S_3=\tau_n\Delta\frac{1}{\lambda_n}+\frac{\tau_n}{(n+1)\lambda_{n+1}}+\frac{1}{\log n}\sum_{k=3}^{n-1}\tau_k\log k\Delta\frac{1}{\lambda_k}+$$
$$+\frac{1}{(n+1)\lambda_{n+1}\log n}\sum_{k=3}^{n-1}\tau_k\log k+O(1).$$

Thus assumptions 1 and 2 imply that

$$S_3=O\left(\frac{\tau_n}{\lambda_n}\right)+O\left(\frac{\tau_n}{\lambda_n}\right)+\frac{1}{\log n}\sum_{k=3}^{n-1}\tau_k\log kO\left(\frac{1}{k\lambda_k\log k}\right)+$$
$$+\frac{1}{n+1}\sum_{k=3}^{n-1}O(1)+O(1)=O(1),$$

i. e.,

(8) $$S_3=O(1)\quad\text{for}\quad n\to\infty.$$

We have

$$S_4=\frac{1}{\log n}\sum_{k=1}^{n}S_k^{(1)}\Delta^2\left(\frac{1}{\lambda_k}\right)\sum_{\nu=k}^{n}\frac{1}{\nu}-\frac{1}{\log n}\sum_{k=1}^{n}S_k^{(1)}\Delta^2\left[\frac{1}{\lambda_k}\right]\left(\frac{1}{k}+\frac{1}{k+1}\right)$$
$$=T_n^{(1)}-U_n,$$

where $T_n^{(1)}$ and U_n denote the first and the second expression on the right-hand side of the last equality. Thus we have

(9) $$S_4=T_n^{(1)}-U_n.$$

Since

$$\Delta\left\{k\sum_{i=k}^{n}\left(\frac{1}{i}+\frac{1}{i+1}\right)\Delta^2\frac{1}{\lambda_i}\right\}$$
$$=\Delta^2\frac{1}{\lambda_k}+\frac{k}{k+1}\Delta^2\frac{1}{\lambda_k}-\sum_{i=k+1}^{n}\left(\frac{1}{i}+\frac{1}{i+1}\right)\Delta^2\frac{1}{\lambda_i}$$

and assumption 2 implies

$$\Delta\left\{k\sum_{i=k}^{n}\left(\frac{1}{i}+\frac{1}{i+1}\right)\Delta^2\frac{1}{\lambda_i}\right\} = O\left(\frac{1}{k\lambda_k\log k}\right)+\sum_{i=k+1}^{\infty}O\frac{1}{i^2\lambda_i\log i}\Big)$$

$$= O\left(\frac{1}{k\lambda_k\log k}\right)+O\left(\frac{1}{k\lambda_k\log k}\right) = O\left(\frac{1}{k\lambda_k\log k}\right),$$

we have for

$$U_n = \frac{1}{\log n}\sum_{i=k}^{n}\frac{s_k}{k}k\sum_{i=k}^{n}\Delta^2\left[\frac{1}{\lambda_i}\right]\left(\frac{1}{i}+\frac{1}{i+1}\right)$$

$$= \frac{1}{\log n}\Delta^2\left[\frac{1}{\lambda_n}\right]\left(\frac{1}{n}+\frac{1}{n+1}\right)\cdot n\sum_{k=1}^{n}\frac{s_k}{k}+$$

$$+\frac{1}{\log n}\sum_{k=1}^{n-1}\Delta\left\{k\sum_{i=k}^{n}\left(\frac{1}{i}+\frac{1}{i+1}\right)\Delta^2\frac{1}{\lambda_i}\right\}\sum_{\nu=1}^{k}\frac{s_\nu}{\nu},$$

the following estimation:

$$U_n = O\left(\frac{\tau_n}{\lambda_n}\right)+\frac{1}{\log n}\sum_{k=3}^{n-1}O(\lambda_k\log k)O\left(\frac{1}{k\lambda_k\log k}\right)+O(1) = O(1).$$

From assumption 1, formulas (6), (7), (8), (9), (1) and the relation $U_n = O(1)$ it follows that

$$T_n = T_n^{(1)}+O(1) \quad \text{for} \quad n\to\infty,$$

and this is the required result.

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