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ON A KIND OF DISPERSION OF SETS

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All sets considered are contained in the interval $[0,1]$ and are Lebesgue measurable. Given a set E we put $E^n = f_n^{-1}(E)$, where $f_n(x) = nx-[nx]$ and $[a]$ denotes the integral part of a. Thus E^n may be called the *dispersion of order* n of the set E.

A lemma of Ulianov (¹) states: *if*

$$\sum_{n=1}^{\infty} |E_n| = \infty, \tag{1}$$

$|E|$ denoting the Lebesgue measure of E, then there exists a sequence of integers $i_1 < i_2 < \ldots$ such that

$$|\overline{\lim_n} E_n^{i_n}| = 1. \tag{2}$$

G. C. Shindalovski has asked whether (2) holds for every increasing sequence $\{i_n\}$ (²). We shall show that the answer is negative. According to a message submitted to the author by D. Menshov this result has been obtained also by E. P. Dolchenko.

THEOREM 1. *Given an increasing sequence of integers $\{i_n\}$ and an $\varepsilon > 0$, there is a sequence of sets E_n fulfilling* (1) *and such that*

$$|\bigcup_{n=1}^{\infty} E_n^{i_n}| < \varepsilon, \tag{3}$$

$$|\overline{\lim_n} E_n^{i_n}| = 0. \tag{4}$$

Proof. First note that $|E| = |E^i|$, $i = 1, 2, \ldots$, for every set E. If E is periodic in $[0,1]$ and its period is $1/m$ (i. e. $1/m$ is the period of

(¹) П. Л. Ульянов, *Расходящиеся ряды Фурье,* Успехи Математических Наук 14 (1961), p. 61-142, especially p. 125.

(²) Problem put forward in 1961 in Moscow on a seminar on real functions directed by Menshov and Ulianov.

its characteristic function), then for every divisor n of m we have

$$|E| = |f_n(E)|, \tag{5}$$

$$\big(f_n(E)\big)^n = E. \tag{6}$$

If $0 < \varepsilon_k < 1$ $(k = 1, 2, \ldots)$ and

$$\sum_{k=1}^{\infty} \varepsilon_k < \varepsilon, \tag{7}$$

then we choose positive integers m_k so that

$$\sum_{k=1}^{\infty} m_k \varepsilon_k = \infty. \tag{8}$$

Put $\mu_0 = 0$, $\mu_k = \sum_{j=1}^{k} m_j$, $w_k = \prod_{n=\mu_{k-1}+1}^{\mu_k} i_n$. Let A_k be a periodic set in $[0, 1)$ with period $1/w_k$ and

$$|A_k| = \varepsilon_k. \tag{9}$$

For every n there is precisely one k_n such that $\mu_{k_n-1} < n \leqslant \mu_{k_n}$. We put

$$E_n = f_{i_n}(A_{k_n}).$$

Since i_n is a divisor of w_{k_n}, on account of (5) and (9) we have

$$|E_n| = |f_{i_n}(A_{k_n})| = |A_{k_n}| = \varepsilon_{k_n}.$$

Hence, from (8),

$$\sum_{n=1}^{\infty} |E_n| = \sum_{k=1}^{\infty} \sum_{n=\mu_{k-1}+1}^{\mu_k} |E_n| = \sum_{k=1}^{\infty} m_k \varepsilon_k = \infty.$$

Thus (1) is fulfilled.

By (6) we have $E_n^{i_n} = A_{k_n}$. Hence $\bigcup_{n=1}^{\infty} E_n^{i_n} \subset \bigcup_{k=1}^{\infty} A_k$. From (9) and (7) it follows that

$$\Big|\bigcup_{n=1}^{\infty} E_n^{i_n}\Big| \leqslant \sum_{k=1}^{\infty} \varepsilon_k < \varepsilon$$

and so we get (3). Now for any $\eta > 0$ we choose first an r so as to have

$$\sum_{k=r}^{\infty} \varepsilon_k < \eta, \tag{10}$$

the choice being possible owing to (7), and the an m such that $\mu_m > r$. Then

$$\bigcup_{n=\mu_m}^{\infty} E_n^{i_n} \subset \bigcup_{k=r}^{\infty} A_k.$$

This yields $\overline{\lim} E_n^{i_n} \subset \bigcup_{k=r}^{\infty} A_k$. Hence, by (9) and (10) we obtain (4) for an arbitrary η. Theorem 1 is thus proved.

We will now distinguish a class of sequences of sets for which the question of Shindalovski admits a positive answer:

THEOREM 2. *If* $|E_n| > \delta > 0$ *for infinitely many* n, *then* (2) *holds for every increasing sequence* $\{i_n\}$.

Proof. Without loss of generality we may assume that $|E_n| > \delta$ for every n. Given an interval $(\alpha, \beta) \subset [0,1)$ we choose N so as to have $1/i_n \leqslant \beta - \alpha$ for $n > N$. Then

$$[\alpha, \beta) = \bigcup_{j=1}^{k} \left[\alpha + \frac{j-1}{i_n}, \alpha + \frac{j}{i_n}\right) \cup \left[\alpha + \frac{k}{i_n}, \beta\right),$$

where $0 \leqslant \beta - \alpha - k/i_n < 1/i_n$ and $k \geqslant 1$. For every interval $I \subset [0,1)$ of length $1/n$ and every set E we have $|I \cap E^n| = |E|/n$. Hence

$$|E_n^{i_n} \cap [\alpha, \beta)| = \sum_{j=1}^{k} \left|E_n^{i_n} \cap \left[\alpha + \frac{j-1}{i_n}, \alpha + \frac{j}{i_n}\right)\right| + \left|E_n^{i_n} \cap \left[\alpha + \frac{k}{i_n}, \beta\right)\right|$$

$$= |E_n| \frac{k}{i_n} + \left|E_n^{i_n} \cap \left[\alpha + \frac{k}{i_n}, \beta\right)\right|.$$

This implies

$$\frac{|E_n^{i_n} \cap [\alpha, \beta)|}{\beta - \alpha} > \frac{|E_n| \frac{k}{i_n}}{(k+1)/i_n} > \frac{\delta}{2} \quad (n > N).$$

The interval $[\alpha, \beta)$ being arbitrary, we infer that the density of the set $\overline{\lim_n} E_n^{i_n}$ is at least $\delta/2$ at every point in $[0,1)$. So (2) follows from the Lebesgue density theorem.

P 470. Do there exist sequences of sets E_n such that $\lim_n |E_n| = 0$ and (2) holds for every increasing sequence $\{i_n\}$?

Let us notice that the order of divergence of the series $\sum_n |E_n|$ is of no importance as far as condition (2) is concerned. In fact, for any series $\sum a_n = \infty$ we can choose the numbers m_k appearing in the proof of theorem 1 in such a way that the series $\sum_n |E_n| = \sum_n \sum_{j=\mu_{n-}+1}^{\mu_n} |A_j|$ be a majorant of $\sum_n a_n$. Nevertheless (4) does hold.

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