SOME THEOREMS ON INTERPOLATION
BY PERIODIC FUNCTIONS

BY

E. STRZELECKI (WROCŁAW)

A sequence \( \{t_n\} \) of positive numbers is said to have property (P) in a class \( K \) of sequences of real numbers if for every \( \{e_n\} \in K \) there exists a continuous periodic function \( f(t), -\infty < t < \infty \), such that

\[
f(t_n) = e_n \quad \text{for} \quad n = 1, 2, \ldots
\]

The problem is to find conditions on \( \{t_n\} \) implying (P). Some results concerning this problem and related questions can be found in [1]–[4].

In particular Ryll-Nardzewski [3] has shown that the sequence \( \{3^n\} \), but no sequence with \( 0 < t_n \leq C \cdot 2^n \) \( (C \) any constant, \( n = 1, 2, \ldots) \), has the property (P) in the class \( K_2 \) of all sequences taking on values 0 and 1. He asked whether every sequence \( \{t_n\} \) such that

\[
t_{n+1} \geq (s + \beta)t_n \quad \text{for} \quad n = 1, 2, \ldots,
\]

where \( \beta > 0 \) and \( s \) is an arbitrary positive integer, has the property (P) in the class \( K_s \) of all sequences taking on \( s \) different values only. I have shown in [4] that this condition is not necessary. In this note some further results concerning Ryll-Nardzewski’s last question will be presented.

In the proofs we shall use closed intervals \([a_n, b_n]\) \((a_n > 0, b_n - a_n \leq \frac{1}{2})\) satisfying some of the following conditions:

\[
(A_n) \quad \frac{t_n}{t_{n-1}} a_{n-1} \leq a_n < b_n \leq \frac{t_n}{t_{n-1}} b_{n-1} \quad (n > 1),
\]

\[
(B_n) \quad [a_n, b_n] \subseteq \begin{cases} [0, \frac{1}{2}] \pmod{1}, & \text{if } e_n = 0, \\ \left[\frac{1}{2}, 1\right] \pmod{1}, & \text{if } e_n = 1, \\ \end{cases}
\]

\[
(C_n) \quad b_n - a_n = \frac{1}{2}.
\]

For any numbers denoted by \( b_n \) and \( b_{n+m} \) \((m = m(n) > 0 \) integer\) and any \( \gamma = \gamma(n) > 0 \)

\[
(D_{n,n,m}) \text{ means } b_{n+m} \leq \frac{t_{n+m}}{t_n} (b_n - \gamma).
\]
We put
\[ q_n = \frac{t_{n+1}}{t_n} \quad (n = 1, 2, \ldots). \]

The following theorem is a generalization of a result by Mycielski [2]:

**Theorem 1.** Every sequence \( \{t_n\} \) satisfying the condition
\[ q_n = \frac{t_{n+1}}{t_n} \geq 3 \quad \text{for} \quad n = 1, 2, \ldots \]
has the property (P) in the class \( K_2 \).

The proof of Theorem 1 is based on seven lemmas.

**Lemma 1.** In order that a sequence \( \{t_n\} \) has property (P) in the class \( K_2 \) it is sufficient that there exists a sequence of intervals \( [a_n, b_n] \) such that for every \( n = 2, 3, \ldots \) condition \( (A_n) \) and for every \( n = 1, 2, \ldots \) condition \( (B_n) \) hold and \( (D_{n,m}) \) is satisfied with an \( m = m_n \), whereas \( \gamma \) is fixed.

Lemma 1 is proved in [4] (see [4], Lemma 1).

**Lemma 2.** If
\[ q_k \geq \frac{25}{6} \]
and for a given interval \([a_k, b_k]\) condition \( (C_k) \) holds, then there exists an interval \([a_{k+1}, b_{k+1}]\) such that we have \( (A_{k+1}), (B_{k+1}), (C_{k+1}) \) and \( (D_{k,1}) \) with \( \gamma = \frac{1}{50} \).

**Proof.** By \( (C_k) \) we have
\[ d_k = q_k(b_k - a_k) = \frac{1}{2} q_k = \frac{3}{2} + \frac{q_k - 3}{2}. \]

Consequently there exists a closed interval \([a_{k+1}, b_{k+1}]\) fulfilling \( (A_{k+1}), (B_{k+1}) \) and \( (C_{k+1}) \) for which
\[ b_{k+1} \leq q_k b_k \frac{q_k - 3}{2} = q_k \left( b_k + \frac{3}{2q_k} - \frac{1}{2} \right). \]

Hence, according to (1), we obtain
\[ b_{k+1} \leq \frac{t_{k+1}}{t_k} \left( b_k - \frac{1}{50} \right), \quad \text{q. e. d.} \]

We shall say that the interval \([a_n, b_n]\) has the property \( (Q_n) \) if there exists an interval \([a_{n+1}, b_{n+1}]\) for which \( (A_{n+1}), (B_{n+1}) \) and \( (C_{n+1}) \) hold and the inequality
\[ (E_{n+1}) \quad b_{n+1} \leq q_n b_n - \frac{1}{16} \]
is fulfilled. In the opposite case we shall say that the interval \([a_n, b_n]\) has the property \((Q'_n)\).

**Lemma 3.** If for a given interval \([a_k, b_k]\) conditions \((C_k)\) and \((Q_k)\) hold, then there exists an interval \([a_{k+1}, b_{k+1}]\) satisfying \((A_{k+1}), (B_{k+1}), (C_{k+1})\) and \((D'_{k+1})\) with \(\gamma = \frac{1}{50}\).

Proof. In virtue of Lemma 2 it is sufficient to consider the case

\[3 \leq q_k < \frac{25}{8}\]

In this case we take an interval \([a_{k+1}, b_{k+1}]\) fulfilling \((A_{k+1}), (B_{k+1}), (C_{k+1})\) and \((E_{k+1})\). By \((E_{k+1})\) we have

\[b_{k+1} \leq q_k \left( b_k - \frac{1}{16q_k} \right) < q_k \left( b_k - \frac{8}{16 \cdot 25} \right) = q_k \left( b_k - \frac{1}{50} \right), \quad \text{q. e. d.}\]

**Lemma 4.** If \(q_k \geq \frac{25}{8}\) and, for a given interval \([a_k, b_k]\), \((C_k)\) holds, then the interval \([a_k, b_k]\) has the property \((Q_k)\).

The proof is analogous to that of Lemma 2, since we have

\[d_k = q_k (b_k - a_k) \geq \frac{25}{8} \cdot \frac{1}{2} = \frac{3}{2} + \frac{1}{16}\]

**Lemma 5.** Let us suppose that for a given interval \([a_k, b_k]\) conditions \((C_k)\) and \((Q_k)\) hold and let \([a_{k+1}, b_{k+1}]\) denote an interval for which conditions \((A_{k+1}), (B_{k+1}), (C_{k+1})\) are fulfilled. (The existence of such an interval follows immediately from \((C_k)\)). If this interval has property \((Q_{k+1})\), then there exists an interval \([a_{k+2}, b_{k+2}]\) such that the conditions \((A_{k+2}), (B_{k+2}), (C_{k+2})\) and \((D'_{k+2})\) with \(\gamma = \frac{4}{625}\) hold.

Proof. In view of Lemma 3, by \((C_{k+1})\) and \((Q_{k+1})\), there exists an interval \([a_{k+2}, b_{k+2}]\) for which we have \((A_{k+2}), (B_{k+2}), (C_{k+2})\) and \((D'_{k+1})\) with \(\gamma = \frac{1}{50}\). Since \((C_k)\) and \((Q'_k)\) hold, Lemma 4 implies

\[q_k < \frac{25}{8}\]

Hence, according to \((D'_{k+1})\) and \((A_{k+1})\) we obtain

\[b_{k+2} \leq \frac{t_{k+2}}{t_{k+1}} \left( b_{k+1} - \frac{1}{50} \right) \leq \frac{t_{k+2}}{t_{k+1}} \left( q_k b_k - \frac{1}{50} \right) = \frac{t_{k+2}}{t_k} \left( b_k - \frac{1}{50q_k} \right) < \frac{t_{k+2}}{t_k} \left( b_k - \frac{4}{625} \right), \quad \text{q. e. d.}\]

In the same way we prove

**Lemma 6.** If any three given intervals \([a_k, b_k], [a_{k+1}, b_{k+1}], [a_{k+2}, b_{k+2}]\) have the following properties:

1) The interval \([a_k, b_k]\) fulfills conditions \((C_k)\) and \((Q'_k)\),

Colloquium Mathematicum XII, 2

16
2) the interval \([a_{k+1}, b_{k+1}]\) fulfills conditions \((A_{k+1}), (B_{k+1}), (C_{k+1})\) and \((Q_{k+1}')\);  
3) the interval \([a_{k+2}, b_{k+2}]\) fulfills conditions \((A_{k+2}), (B_{k+2}), (C_{k+2})\) and \((Q_{k+2}')\),
then there exists an interval \([a_{k+3}, b_{k+3}]\) for which we have \((A_{k+3}), (B_{k+3}), (C_{k+3})\) and \((D_{k+3}')\) with \(\gamma = 2^5/5^6\).

Now we proceed to the main lemma.

**Lemma 7.** If for a given interval \([a_k, b_k]\) condition \((C_k)\) holds, then at least one of the following possibilities takes place:

1° there exists an interval \([a_{k+1}, b_{k+1}]\) such that conditions \((A_{k+1})\), \((B_{k+1})\), \((C_{k+1})\) and \((D_{k+1}')\) with \(\gamma \geq 2^5/5^6\) are fulfilled,

2° there exist two intervals \([a_{k+1}, b_{k+1}]\) and \([a_{k+2}, b_{k+2}]\) such that conditions \((A_{k+1}), (B_{k+1}), (A_{k+2}), (B_{k+2}), (C_{k+2})\) and \((D_{k+3}')\) with \(\gamma \geq 2^5/5^6\) are fulfilled,

3° there exist three intervals \([a_{k+l}, b_{k+l}]\), \(l = 1, 2, 3\), such that conditions \((A_{k+l})\), \((B_{k+l})\), \(l = 1, 2, 3\), \((C_{k+3})\) and \((D_{k+3}')\) with \(\gamma \geq 2^5/5^6\) are fulfilled.

**Proof.** If \((Q_k)\) holds, then it follows from Lemma 3 that the first possibility occurs. Let us assume that \((Q_{k}')\) holds. Since \(b_k - a_k = \frac{1}{2}\) and \(q_k \geq 3\), there exists an interval \([v_{k+1}, v_{k+1}]\) satisfying \((A_{k+1}), (B_{k+1})\) and \((C_{k+1})\). Hence, by \((Q_{k}')\), we obtain

\[
(E_{k+1}') \quad \quad v_{k+1} > q_k b_k - \frac{1}{16}.
\]

We first show that

\[
(2) \quad (v_{k+1} - 1) - q_k a_k < \frac{1}{2}.
\]

In fact, if the contrary is true, then according to \((A_{k+1})\) we obtain

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
 & a_k & b_k & v_{k+1} & v_{k+1} & \frac{1}{2} & u_{k+1} & \frac{1}{2} & v_{k+1} & q_k b_k \\
\hline
\end{array}
\]

Fig. 1

\[
q_k a_k \leq v_{k+1} - \frac{3}{2} < v_{k+1} - 1 \leq q_k b_k - 1.
\]

Moreover,

\[
[v_{k+1} - \frac{3}{2}; v_{k+1} - 1] = [u_{k+1}, v_{k+1}] \mod 1.
\]

Consequently the interval \([v_{k+1} - \frac{3}{2}; v_{k+1} - 1]\) satisfies \((A_{k+1}), (B_{k+1}), (C_{k+1})\) and \((E_{k+1})\), which contradicts \((Q_{k}').\)
We also note that
\[ q_k a_k < v_{k+1} - 1. \]

This inequality immediately follows from \((E_{k+1}^*)\), \(q_k \geq 3\) and \(b_k = a_k + \frac{1}{2}\). By (2) and (3),
\[ [q_k a_k, (v_{k+1} - 1)] \subset [u_{k+1}, v_{k+1}] \mod 1, \]
i.e., the interval \([q_k a_k, (v_{k+1} - 1)]\) fulfills conditions \((A_{k+1})\) and \((B_{k+1})\) (but not \((C_{k+1})\)).

If the interval \([u_{k+1}, v_{k+1}]\) has property \((Q_{k+1})\), then in virtue of Lemma 5 the second possibility is realized by putting \(a_{k+1} = u_{k+1}, b_{k+1} = v_{k+1}\).

Now let us suppose that for the interval \([u_{k+1}, v_{k+1}]\) the property \((Q'_{k+1})\) holds. We shall denote by \([u_{k+2}, v_{k+2}]\) an interval satisfying conditions \((A_{k+2}), (B_{k+2}), (C_{k+2})\) and \((B'_{k+2})\). Repeating the former considerations we show that the interval \([q_{k+1} u_{k+1}, (v_{k+2} - 1)]\) also satisfies \((A_{k+2})\) and \((B_{k+2})\) (see fig. 2).

We shall now prove that
\[ q_{k+1}(q_k a_k) < v_{k+2} - \frac{7}{2} < q_{k+1}(v_{k+1} - 1). \]

We remind that for the intervals \([a_k, b_k], [u_{k+1}, v_{k+1}], [u_{k+2}, v_{k+2}]\) conditions \((C_k), (A_{k+l}), (B_{k+l}), (C_{k+l}) (l = 1, 2), (Q_k)\) and \((Q'_{k+1})\) hold. Consequently, by the definition of \((Q'_{m}), (E_{k+1}'), (E_{k+2}')\) are also true. Hence
\[ v_{k+2} > q_{k+1} v_{k+1} - \frac{1}{16} > q_{k+1}(q_k b_k - \frac{1}{16}) - \frac{1}{16} \]
\[ = q_{k+1} q_k (a_k + \frac{1}{2}) - \frac{1}{16} (q_{k+1} + 1) \geq q_{k+1}(q_k a_k) + \frac{9}{2} - \frac{1}{16}(q_{k+1} + 1). \]

In virtue of Lemma 4 it follows from \((C_k)\) and \((Q'_{k+1})\) that
\[ (F_{k+1}) \quad q_{k+1} < \frac{25}{8}. \]

Therefore we obtain
\[ v_{k+2} - \frac{7}{2} > q_{k+1}(q_k a_k) + 1 - \frac{33}{128} > q_{k+1}(q_k a_k). \]
From \((A_{k+2})\) and \((F_{k+1})\) it follows that
\[
v_{k+2} - \frac{7}{2} \leq q_{k+1} v_{k+1} - \frac{7}{2} < q_{k+1} v_{k+1} - q_{k+1}.
\]
Hence, by (6), inequality (5) holds.

If \(v_{k+2}\) satisfies the inequality
\[(G_{k+2}) \quad v_{k+2} - 3 \leq q_{k+1} (v_{k+1} - 1)\]
(see fig. 2), then we shall put
\[
a_{k+1} = q_{k} a_{k}, \quad a_{k+2} = u_{k+2} - 3,
\]
\[
b_{k+1} = v_{k+1} - 1, \quad b_{k+2} = v_{k+2} - 3.
\]

As we have already noticed, the interval \([q_{k} a_{k}, (v_{k+1} - 1)]\) satisfies conditions \((A_{k+1})\) and \((B_{k+1})\) in virtue of (4). From (5) and \((G_{k+2})\) it follows that for the interval \([u_{k+2} - 3, v_{k+2} - 3]\) the conditions \((A_{k+2})\), \((B_{k+2})\) and \((C_{k+2})\) hold (see fig. 2). Moreover, by \((A_{k+1})\) and \((F_{k})\),
\[
b_{k+1} = v_{k+1} - 1 \leq q_{k} b_{k} - 1 = q_{k} \left( b_{k} - \frac{1}{q_{k}} \right) = \frac{t_{k+1}}{t_{k}} \left( b_{k} - \frac{8}{25} \right),
\]
i.e., the condition \((D'_{k,1})\) with \(\gamma = \frac{8}{25}\) is satisfied. Thus, the second condition of the assertion is realized in the case when \((Q'_{k})\), \((Q'_{k+1})\) and \((G_{k+2})\) hold.

Now let us assume that \((Q'_{k})\), \((Q'_{k+1})\) and
\[(G'_{k+2}) \quad v_{k+2} - 3 > q_{k+1} (v_{k+1} - 1)\]
hold. In this case putting \(a_{k+1} = q_{k} a_{k}\), \(b_{k+1} = v_{k+1} - 1\) we infer by (5) that the interval \([v_{k+2} - \frac{7}{2}, q_{k+1} (v_{k+1} - 1)]\) satisfies the conditions \((A_{k+2})\) and \((B_{k+2})\) (see fig. 3). We shall consider the intervals with the index

\[n = k + 3.\]

If the interval \([u_{k+2}, v_{k+2}]\) has the property \((Q_{k+2})\), then the third condition of the assertion follows from Lemma 6. (In this case we put \(a_{k+1} = u_{k+1}\), \(b_{k+1} = v_{k+1}\), \(a_{k+2} = u_{k+2}\), \(b_{k+2} = v_{k+2}\).)
Let us suppose that for \([u_{k+2}, v_{k+2}]\) the property \((Q_{k+2}')\) holds. Denote by \([u_{k+3}, v_{k+3}]\) the interval satisfying conditions \((B_{k+3})\), \((C_{k+3})\) and \((A_{k+3})\):

\[
q_{k+2} u_{k+2} \leq u_{k+3} < v_{k+3} \leq q_{k+2} v_{k+2}.
\]

If \((G_{k+3})\) holds we put:

\[
a_{k+1} = u_{k+1}, \quad a_{k+2} = q_{k+1} u_{k+1}, \quad a_{k+3} = u_{k+3} - 3, \\
b_{k+1} = v_{k+1}, \quad b_{k+2} = v_{k+2} - 1, \quad b_{k+3} = v_{k+3} - 3.
\]  \(7'\)

The interval \([a_{k+2}, b_{k+2}]\) fulfills \((A_{k+2})\) and \((B_{k+2})\) which can be shown in the same way as the conditions \((A_{k+1})\) and \((B_{k+1})\) have been checked for the interval \([q_k a_k, (v_{k+1} - 1)]\) under assumption \((Q_k')\). Further, the interval \([a_{k+3}, b_{k+3}]\) satisfies \((A_{k+3})\). We obtain this from \((Q_{k+1}')\), \((Q_{k+2}')\) and \((G_{k+3})\) similarly as \((A_{k+2})\) has been proved for the interval \([u_{k+2} - 3, v_{k+2} - 3]\) under assumption \((Q_k')\), \((Q_{k+1}')\) and \((G_{k+2})\). Finally, \([a_{k+3}, b_{k+3}]\) satisfies \((B_{k+3})\) and \((C_{k+3})\), since \([u_{k+3}, v_{k+3}]\) does. Taking into account that the intervals \([u_{k+1}, v_{k+1}]\) and \([u_{k+2}, v_{k+2}]\) satisfy \((A_{k+1})\) and \((A_{k+2})\), we obtain from \((F_k)\) and \((F_{k+1})\):

\[
b_{k+2} = v_{k+2} - 1 \leq q_{k+1} v_{k+1} - 1 \leq q_{k+1} q_k b_k - 1
\]

\[
\leq q_{k+1} q_k \left(b_k - \frac{64}{625}\right) = \frac{t_{k+2}}{t_k} \left(b_k - \frac{64}{625}\right),
\]

i.e., the condition \((D_{k+2}')\) with \(\gamma = \frac{64}{625}\) is also fulfilled. Hence, in the case that \((Q_k'), (Q_{k+1}'), (Q_{k+2}')\) and \((G_{k+3})\) hold, the third possibility of the assertion is again realized.

Now let us suppose that:

\[
(G_{k+3}') \quad v_{k+3} - 3 > q_{k+2} (v_{k+2} - 1).
\]

We shall prove that the inequality:

\[
q_{k+2} \left(v_{k+2} - \frac{7}{2}\right) \leq v_{k+3} - \frac{21}{2} < v_{k+3} - 1 \leq q_{k+2} q_{k+1} (v_{k+1} - 1)
\]  \(8\)

follows from \((Q_k'), (Q_{k+1}'), (Q_{k+2}')\) and \((G_{k+3}')\). Taking \((G_{k+3}')\) into account we obtain:

\[
v_{k+3} - \frac{21}{2} > q_{k+2} (v_{k+2} - 1) - \frac{15}{2}
\]

\[
\geq q_{k+2} (v_{k+2} - 1) - \frac{q_{k+2}}{3} \cdot \frac{15}{2} = q_{k+2} \left(v_{k+2} - \frac{7}{2}\right).
\]

Thus the first inequality in \(8\) is proved. Since \((C_{k+2})\) and \((Q_{k+2}')\) hold, we have, by Lemma 4,

\[
(F_{k+2}) \quad q_{k+2} < \frac{25}{8}.
\]
From \((A_{k+3}), (A_{k+2}), (F_{k+1})\) and \((F_{k+2})\) it follows that

\[
v_{k+3} - 10 \leq q_{k+2}v_{k+2} - 10 \leq q_{k+2}q_{k+1}v_{k+1} - 10
\]

\[
< q_{k+2}q_{k+1}v_{k+1} - q_{k+2}q_{k+1} = q_{k+2}q_{k+1}(v_{k+1} - 1),
\]

i.e., the third inequality in (8) is also true. Hence, if we put

\[
\begin{align*}
ak_{l+1} &= q_{k}a_{k}, \\ak_{l+2} &= q_{k+2} - \frac{7}{2}, \\ak_{l+3} &= q_{k+3} - 10, \\bk_{l+1} &= v_{k+1} - 1, \\bk_{l+2} &= q_{k+1}(v_{k+1} - 1), \\bk_{l+3} &= v_{k+3} - 10,
\end{align*}
\]

the intervals \([a_{k+1}, b_{k+1}], l = 1, 2, 3\), satisfy conditions \((A_{k+3}), (B_{k+1}), l = 1, 2, 3\), respectively and \([a_{k+3}, b_{k+3}]\) fulfills \((C_{k+3})\). Moreover, by \((A_{k+1})\) and \((F_{k})\) we obtain

\[
b_{k+1} = v_{k+1} - 1 \leq q_{k}b_{k} - 1 \leq \frac{t_{k+1}}{t_{k}} \left(b_{k} - \frac{8}{25}\right).
\]

Therefore \((D_{k,1}')\) with \(\gamma > 2^{5}/5^{6}\) is also satisfied.

The reader may note that the following cases have been considered in the above-mentioned proof:

1) \((Q_{k})\);

2) \((Q_{k}', (Q_{k+1})\);

3) \((Q_{k}'), (Q_{k+1}'), (G_{k+2})\);

4) \((Q_{k}'), (Q_{k+1}), (G_{k+2}), (Q_{k+2})\);

5) \((Q_{k}'), (Q_{k+1}'), (G_{k+2}), (Q_{k+2}'), (G_{k+3})\);

6) \((Q_{k}'), (Q_{k+1}'), (G_{k+2}), (Q_{k+2}'), (G_{k+3})\).

Thus all possibilities have been exhausted and Lemma 7 is proved.

Proof of Theorem 1. Let us put \(a_{1} = 1, b_{1} = \frac{3}{2}\), if \(e_{1} = 0\) and \(a_{1} = \frac{1}{2}, b_{1} = 1\), if \(e_{1} = 1\) and build the sequence of intervals \([a_{n}, b_{n}]\), \(n = 1, 2, \ldots\), by induction as follows: if \(b_{k} - a_{k} = \frac{1}{2}\), then we choose subsequent intervals according to Lemma 7 till we get an interval of length \(\frac{1}{2}\). On account of Lemma 1 in order to prove Theorem 1 it is sufficient to show that condition \((D_{n,m}')\) is satisfied for every \(n\) with a \(\gamma > 0\) independent of \(n\), e.g. with \(\gamma = 2^{11}/5^{10}\).

1) If \(b_{n} - a_{n} = \frac{1}{2}\), then we obtain \((D_{n,m}')\) \((\gamma > 2^{5}/5^{6}, m = 1, 2\) or 3) as a direct consequence of Lemma 7 putting \(n = k\).

2) If \(b_{n} - a_{n} < \frac{1}{2}\) then, by Lemma 7, it is sufficient to distinguish the following cases:

a) Case \((Q_{n-1}'), (Q_{n}'), (G_{n+1})\). Then the intervals \([a_{n}, b_{n}]\) and \([a_{n+1}, b_{n+1}]\) satisfy formulae (7) with \(n = k + 1\) or (7') with \(n = k + 2\). Since \(b_{n+1} - a_{n+1} = \frac{1}{2}\) there exists, by Lemma 7, a positive integer \(m\)
(m = 1, 2 or 3) such that

\[(D'_{n+1,m}) \quad b_{n+1+m} \leq \frac{t_{n+1+m}}{t_{n+1}} \left( b_{n+1} - \frac{2^5}{5^6} \right) \].

Consequently, according to \((A_{n+1})\) and \((F_n)\) we obtain

\[b_{n+1+m} \leq \frac{t_{n+1+m}}{t_{n+1}} \left( q_n b_n - \frac{2^5}{5^6} \right) \leq \frac{t_{n+1+m}}{t_n} \left( b_n - \frac{2^8}{5^6} \right)\].

Thus \((D'_{n,m+1})\) with \(\gamma > 2^{11}/5^{10}\), \(m = 1, 2\) or \(3\) holds.

b) Case \((Q'_{n-1})\), \((Q'_n)\), \((Q'_{n+1})\), \((Q'_{n+2})\). The intervals \([a_n, b_n]\), \([a_{n+1}, b_{n+1}]\) and \([a_{n+2}, b_{n+2}]\) satisfy formulae (9) with \(n = k+1\). In this case \(b_n - a_n < \frac{1}{2}\), \(q_n < \frac{25}{8}\), \(b_{n+1} - a_{n+1} < \frac{1}{2}\), \(q_{n+1} < \frac{25}{8}\), \(b_{n+2} - a_{n+2} = \frac{1}{2}\). Reasoning like in the previous case we get the inequality

\[b_{n+m} \leq \frac{t_{n+m}}{t_n} \left( b_n - \frac{2^{11}}{5^{10}} \right)\]

for a positive integer \(m\) \((2 \leq m \leq 5)\).

c) Case \((Q'_{n-2})\), \((Q'_{n-1})\), \((Q'_{n})\), \((Q'_{n+1})\), \((Q'_{n+1})\). The intervals \([a_n, b_n]\) and \([a_{n+1}, b_{n+1}]\) fulfill formulae (9) with \(n = k+2\). In this case \(b_n - a_n < \frac{1}{2}\), \(q_n < \frac{25}{8}\), \(b_{n+1} - a_{n+1} = \frac{1}{2}\). Therefore, by Lemma 7, there exists a positive integer \(m\) \((1 \leq m \leq 4)\) such that

\[b_{n+m} < \frac{t_{n+m}}{t_n} \left( b_n - \frac{2^8}{5^6} \right)\].

Theorem 1 is thus proved.

In a similar way we can prove the following theorems.

**Theorem 2.** The sequence

\[t_n = \left( 3 - \frac{1}{s+1} \right)^n \quad (n = 1, 2, 3, \ldots),\]

where \(s\) is any positive integer, has the property \(P\) in the class \(K_2\).

Remark. The case \(\{(5/2)^n\}\) must be considered separately.

**Theorem 3.** If the \(q_n\)'s assume only two integer values \(m\) or \(m+1\) \((m \geq 2)\) and the number of consecutive terms \(q_n = m\) does not exceed a finite upper bound, then the sequence \(\{t_n\}\) has the property \(P\) in the class \(K_m\).

Remark. In the case of \(m = 2\) the period \(\delta\) of the function \(f(t)\) for which \(f(t_n) = \varepsilon_n\) \((\varepsilon_n = 0, 1)\) can be obtained by the formula

\[\frac{1}{\delta} = \sum_{n=1}^{\infty} \frac{a_n}{t_n},\]
where \( a_1 = \frac{1}{2} \epsilon_1 \) and for \( n = 1, 2, \ldots \)

\[
a_{n+1} = \begin{cases} 
\frac{1}{2} \epsilon_{n+1}, & \text{if } q_n = 2, \\
\frac{1}{2} |\epsilon_{n+1} - \epsilon_n|, & \text{if } q_n = 3.
\end{cases}
\]

REFERENCES


DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY, WROCŁAW

*Reçu par la Rédaction le 15. 6. 1963*