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CONCERNING ALMOST PERIODIC EXTENSIONS OF FUNCTIONS

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One says that a set Z of real numbers has the property $\boldsymbol{I}_0$ ($Z \in \boldsymbol{I}_0$) if

(1) every real and bounded function defined on Z can be extended to a uniformly almost periodic (u. a. p.) function on the real line.

More detailed informations on the property $\boldsymbol{I}_0$ and other related notions can be found in [1]. Let us reproduce here only two simple propositions concerning $\boldsymbol{I}_0$ (cf. [1], p. 25):

PROPOSITION 1. *Sets Z with property $\boldsymbol{I}_0$ are either finite or denumerable and discrete.*

PROPOSITION 2. *In the definition* (1) *of the class $\boldsymbol{I}_0$ it suffices to assume that only functions taking on two values only* (*e. g. zero or one*) *on Z admit u. a. p. extensions.*

We will prove the following property of the class $\boldsymbol{I}_0$:

THEOREM 1. *If $Z \in \boldsymbol{I}_0$ and F is a finite set of real numbers, then $Z \cup F \in \boldsymbol{I}_0$.*

Proof. Obviously we can restrict ourselves to the case when the set Z is denumerable, F is a one-point set $\{z_0\}$ and $z_0 \notin Z$. Let us observe that it suffices to prove the existence of a u. a. p. function $e(t)$ such that

$$(2) \qquad e(z_0) = 1 \quad \text{and} \quad e(z) = 0 \quad \text{for all} \quad z \in Z.$$

In fact, in such a case the formula

$$(3) \qquad f = f(z_0)e + (1-e)\overline{(f|_Z)},$$

where $\overline{(f|_Z)}$ denotes a u. a. p. extension of the restriction to Z of a bounded and continuous function f given on the set $Z \cup \{z_0\}$, yields a u. a. p. extension of f.

Let us assume now for a moment that (2) fails, i. e., the following uniqueness condition holds:

(4) If f_1 and f_2 are u. a. p. functions and $f_1(z) = f_2(z)$ for all $z \in Z$, then $f_1(z_0) = f_2(z_0)$.

We will show that (4) produces a contradiction. On the ring $B(Z)$ of all real and bounded functions on Z we define the functional φ:

(5) $\varphi(g) = \bar{g}(z_0)$, where $\bar{g}$ is a u. a. p. extension of $g \in B(Z)$.

In view of (4) the value $\varphi(g)$ is well defined. It is easy to see that φ is a homomorphism of the ring $B(Z)$ onto the ring of real numbers. The ring $B(Z)$ can be treated as a subset of the topological space R^Z (= the Cartesian product of $\aleph_0$ copies of the real line R). The diagram of φ is an analytical set. In fact, we can write:

$$\{\langle g, y\rangle : \varphi(g) = y\} = \bigvee_{\bar{g}} \{\langle g, \bar{g}, y\rangle : \bar{g} \in \text{u. a. p. and } \bar{g}\,|_Z = g \text{ and } \bar{g}(z_0) = y\},$$

and it is easy to see that the set in the brackets is a Borel subset of the Cartesian product $B(Z) \times C(R) \times R$, where $C(R)$ denotes the space of real and continuous functions on R. The analyticity of the diagram of φ implies that φ is a Borel functional, i. e., it is measurable with respect to the field of Borel subsets of $B(Z)$ (see [2], p. 398).

Now the following theorem of Sierpiński can be directly applied (see [3]; our formulation differs slightly from the original one):

THEOREM (Sierpiński). *Every Borel homomorphism ψ from $B(N)$ (N is an arbitrary denumerable set) onto R is trivial, i. e., it is determined by a point $n_1 \in N$:*

$$\psi(g) = g(n_1) \quad \textit{for all} \quad g \in B(N).$$

Hence there exists a number $z_1 \in Z$ such that $\varphi(g) = \bar{g}(z_0) = g(z_1)$ for all $g \in B(Z)$, which immediately yields a contradiction, since there are u. a. p. functions separating the points z_0 and z_1. In this way the proof of the existence of a u. a. p. function e satisfying (2) is completed.

Remark 1. Theorem 1 remains valid if we replace in its formulation the property $\boldsymbol{I}_0$ by the property $\boldsymbol{I}$ (cf. [1]); $Z \in \boldsymbol{I}$ means that every bounded and uniformly continuous function on Z has a u. a. p. extension. In this way we obtain

THEOREM 1. *If $Z \in \boldsymbol{I}$ and F is a finite set of real numbers, then $Z \cup F \in \boldsymbol{I}$.*

Remark 2. In [1] the following conjecture was stated as a part of P 452:

(H) If $Z \in \boldsymbol{I}$, then the weak closure of Z is of the Haar measure zero.

The Haar measure and weak closure are understood with respect the so-called Bohr compactification of the real line (cf. [1], p. 24).

By methods similar to those presented above we can easily reduce (H) to the following conjecture:

(C) Every linear, non-negative and Borel functional ψ defined on $B(N)$ has the form

$$\psi(g) = \sum_{n \epsilon N} \lambda_n g(n) \quad \text{for all} \quad g \epsilon B(N),$$

where $\lambda_n \geqslant 0$ and $\sum_{n \epsilon N} \lambda_n < \infty$.

It is easy to see that (C) implies the theorem of Sierpiński. The author is rather inclined to believe that (C) holds.

Remark 3. Finally, let us mention that every functional ψ (see (C)) can be written in the integral form

$$\psi(g) = \int_N g(n) d\mu \quad \text{for} \quad g \epsilon B(N),$$

where μ is a finitely additive and non-negative measure defined on the set 2^N of all subsets of N, and the correspondence between ψ and μ is one-to-one. The set 2^N will be considered as a topological space with the product-topology (2^N is homeomorphic with the Cantor dyadic set). Now we are able to reformulate the conjecture (C):

(C′) Every finitely additive, non-negative, Borel measurable measure μ on 2^N vanishing on finite sets is identically equal to zero.

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[2] C. Kuratowski, *Topologie I*, Warszawa-Wrocław 1948.

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