CONCERNING ALMOST PERIODIC EXTENSIONS OF FUNCTIONS

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One says that a set $Z$ of real numbers has the property $I_0$ ($Z \in I_0$) if
(1) every real and bounded function defined on $Z$ can be extended
to a uniformly almost periodic (u. a. p.) function on the real line.

More detailed informations on the property $I_0$ and other related no-
tions can be found in [1]. Let us reproduce here only two simple propositions
concerning $I_0$ (cf. [1], p. 25):

PROPOSITION 1. Sets $Z$ with property $I_0$ are either finite or denumerable
and discrete.

PROPOSITION 2. In the definition (1) of the class $I_0$ it suffices to assume
that only functions taking on two values only (e. g. zero or one) on $Z$ admit
u. a. p. extensions.

We will prove the following property of the class $I_0$:

THEOREM 1. If $Z \in I_0$ and $F$ is a finite set of real numbers, then $Z \cup F \in I_0$.

Proof. Obviously we can restrict ourselves to the case when the set $Z$
is denumerable, $F$ is a one-point set $\{z_0\}$ and $z_0 \notin Z$. Let us observe that
it suffices to prove the existence of a u. a. p. function $e(t)$ such that

(2) $e(z_0) = 1$ and $e(z) = 0$ for all $z \in Z$.

In fact, in such a case the formula

(3) $f = f(z_0)e + (1 - e)(f|_Z)$,

where $(f|_Z)$ denotes a u. a. p. extension of the restriction to $Z$ of a bounded
and continuous function $f$ given on the set $Z \cup \{z_0\}$, yields a u. a. p.
extension of $f$.

Let us assume now for a moment that (2) fails, i. e., the following
uniqueness condition holds:

(4) If $f_1$ and $f_2$ are u. a. p. functions and $f_1(z) = f_2(z)$ for all $z \in Z$,
then $f_1(z_0) = f_2(z_0)$.
We will show that (4) produces a contradiction. On the ring \( B(Z) \) of all real and bounded functions on \( Z \) we define the functional \( \varphi \):

\[ \varphi(g) = \bar{g}(z_0), \text{ where } \bar{g} \text{ is a u. a. p. extension of } g \in B(Z). \]

In view of (4) the value \( \varphi(g) \) is well defined. It is easy to see that \( \varphi \) is a homomorphism of the ring \( B(Z) \) onto the ring of real numbers. The ring \( B(Z) \) can be treated as a subset of the topological space \( R^Z \) (= the Cartesian product of \( \aleph_0 \) copies of the real line \( R \)). The diagram of \( \varphi \) is an analytical set. In fact, we can write:

\[ \{ \langle g, y \rangle : \varphi(g) = y \} = \bigvee_{\bar{g}} \{ \langle g, \bar{g}, y \rangle : \bar{g} \text{ u. a. p. and } \bar{g}|_Z = g \text{ and } \bar{g}(z_0) = y \}, \]

and it is easy to see that the set in the brackets is a Borel subset of the Cartesian product \( B(Z) \times C(R) \times R \), where \( C(R) \) denotes the space of real and continuous functions on \( R \). The analyticity of the diagram of \( \varphi \) implies that \( \varphi \) is a Borel functional, i. e., it is measurable with respect to the field of Borel subsets of \( B(Z) \) (see [2], p. 398).

Now the following theorem of Sierpiński can be directly applied (see [3]; our formulation differs slightly from the original one):

**Theorem (Sierpiński).** Every Borel homomorphism \( \varphi \) from \( B(N) \) (\( N \) is an arbitrary denumerable set) onto \( R \) is trivial, i. e., it is determined by a point \( n_1 \in N \):

\[ \varphi(g) = g(n_1) \text{ for all } g \in B(N). \]

Hence there exists a number \( z_1 \in Z \) such that \( \varphi(g) = \bar{g}(z_0) = \bar{g}(z_1) \) for all \( g \in B(Z) \), which immediately yields a contradiction, since there are u. a. p. functions separating the points \( z_0 \) and \( z_1 \). In this way the proof of the existence of a u. a. p. function \( e \) satisfying (2) is completed.

Remark 1. Theorem 1 remains valid if we replace in its formulation the property \( I_0 \) by the property \( I \) (cf. [1]); \( Z \in I \) means that every bounded and uniformly continuous function on \( Z \) has a u. a. p. extension. In this way we obtain

**Theorem 1.** If \( Z \in I \) and \( F \) is a finite set of real numbers, then \( Z \sim F \in I \).

Remark 2. In [1] the following conjecture was stated as a part of P 452:

(H) If \( Z \in I \), then the weak closure of \( Z \) is of the Haar measure zero.

The Haar measure and weak closure are understood with respect the so-called Bohr compactification of the real line (cf. [1], p. 24).

By methods similar to those presented above we can easily reduce (H) to the following conjecture:
(C) Every linear, non-negative and Borel functional \( \psi \) defined on \( B(N) \) has the form

\[
\psi(g) = \sum_{n \in N} \lambda_n g(n) \quad \text{for all } g \in B(N),
\]

where \( \lambda_n \geq 0 \) and \( \sum_{n \in N} \lambda_n < \infty \).

It is easy to see that (C) implies the theorem of Sierpiński. The author is rather inclined to believe that (C) holds.

**Remark 3. Finally, let us mention that every functional \( \psi \) (see (C)) can be written in the integral form**

\[
\psi(g) = \int_N g(n) \mathrm{d}\mu \quad \text{for } g \in B(N),
\]

where \( \mu \) is a finitely additive and non-negative measure defined on the set \( 2^N \) of all subsets of \( N \), and the correspondence between \( \psi \) and \( \mu \) is one-to-one. The set \( 2^N \) will be considered as a topological space with the product-topology (\( 2^N \) is homeomorphic with the Cantor dyadic set). Now we are able to reformulate the conjecture (C):

(C') Every finitely additive, non-negative, Borel measurable measure \( \mu \) on \( 2^N \) vanishing on finite sets is identically equal to zero.

**REFERENCES**


*Reçu par la Rédaction le 15. 9. 1963*