COLLOQUIUM MATHEMATICUM

VOL. XII 1964 FASC. 1

DEGREE OF CONVERGENCE OF THE EXTREMAL POINTS METHOD FOR DIRICHLET'S PROBLEM IN THE SPACE

BY

W. KLEINER (CRACOW)

PRELIMINARIES AND SYMBOLS

1. This paper presents an application of general considerations in [9]. Thus we shall omit here all explanation of notions described there. Nevertheless, all suppositions and conventional symbols are explicitly mentioned below. The problem and result is ultimately formulated in section 7.

2. Let $x = [x^1, \ldots, x^N]$ be the variable point in the N-dimensional Cartesian linear space R^N $(N \geqslant 3)$, $|x| = \left(\Sigma (x^i)^2\right)^{1/2}$ the usual norm, $|x-y|$ the distance from x to $y \in R^N$. The (Alexandroff's) infinity ∞ is not included into R^N. In particular, A^- (the closure of a set A) does not contain ∞, even for unbounded A.

For any two sets $A, B \subset R^N$, their distance is $\varrho(A, B) = \inf\{|x-y|: x \in A, y \in B\}$; $\varrho(x, B) = \varrho(\{x\}, B)$. *For any positive number* δ, $\delta A = \{x: \varrho(x, A) < \delta\}$.

We consider signed measures $\mu, \sigma, \tau, \ldots$ on R^N ([1], p. 233). μ^+ is the positive, μ^- the negative part of μ: $\mu = \mu^+ - \mu^-$. The variation $|\mu| = \mu^+ + \mu^-$ is itself a measure. $\mu(R^N)$ is the mass of μ, $|\mu|(R^N)$ its absolute mass. $\mu \geqslant 0$ means that μ is a positive measure; $|\mu| \leqslant \nu$ means $\nu - |\mu| \geqslant 0$. $\mu \mid A$ is the trace of μ on A: $\int f d\mu \mid A = \int_A f d\mu = \int f \varphi_A d\mu$ (φ_A — characteristic function of A). $\mu \subset A$ *means* $\mu = \mu \mid A$.

We will be mainly concerned with the following classes of measures on a compact set F to be described later:

$$T = \{\mu: \mu \subset F,\ \mu \geqslant 0, \mu(F) = 1\}, \tag{2.1}$$

$$T_n = \left\{\tau: \tau = \sum_{j=1}^{n} \frac{1}{n} \varepsilon(a_j),\ \{a_1, \ldots, a_n\} \subset F\right\}, \tag{2.2}$$

where $\varepsilon(a)$ is the unit measure at a, i.e. $\int h(x)\,d\varepsilon(a) = h(a)$ for every continuous (or even semicontinuous) h.

3. For any measures σ, τ we put

$$(3.1)\qquad U^\sigma(x) = \int |x-y|^{2-N} d\sigma_y, \quad (\sigma, \tau) = \int U^\sigma d\tau, \quad \|\sigma\|^2 = (\sigma, \sigma),$$

$$(3.2)\qquad U_0^\tau(x) = \int |x-y|_0^{2-N} d\tau_y, \quad (\tau, \sigma)_0 = \int U_0^\tau d\sigma, \quad \|\tau\|_0^2 = (\tau, \tau)_0,$$

where the index 0 indicates that for $x = y$ the integrand is to be replaced by 0. All these expressions are to be used only if well-defined. U^σ is the Newton's *potential.* The *energy* $\|\sigma\|^2$ is positive if $\sigma \neq 0$ [4]. The measures with finite energy form a *pre-Hilbert space* $\mathscr{E}$ *with the inner product* (σ, τ) [2]. If $T \cap \mathscr{E}_0$ is non-void, there exists [4] an *equilibrium measure* $\eta \in T$ characterized by

$$(3.3)\qquad U^\eta(x) = \|\eta\|^2 \leqslant \|\mu\|^2 \quad (x \in F,\ \mu \in T);$$

the first equality may fail in some points of F, but this is never the case for F as in section 4.

Formulae (3.2) are destined for discrete measures, but a continuos one is by no way excluded. We will use them 1° for $\sigma, \tau \in T_n$, 2° for U^σ continuous, $\tau \in T_n$ and 3° for $\sigma, \tau \in \mathscr{E}$. It is immediately seen that, for 2° and 3°, $(\sigma, \tau)_0 = (\sigma, \tau)$, and in all cases mentioned $(\sigma, \tau)_0$ *is a symmetric bilinear form.* From the above cases, we can extend it linearly. To facilitate the lecture, we write down some explanation formulae:

$$(3.4)\qquad \big(\varepsilon(a), \varepsilon(b)\big)_0 = \begin{cases} |a-b|^{2-N} & \text{if} \quad a \neq b, \\ 0 & \text{if} \quad a = b, \end{cases}$$

$$\|\tau\|_0^2 = \sum_{i \neq k}\sum n^{-2} |a_i - a_k|^{2-N} \quad (\tau \in T_n \text{ defined by (2.2)}),$$

$$\begin{aligned}\|\tau-\varphi\|_0^2 &= \|\tau\|_0^2 - 2(\varphi, \tau)_0 + \|\varphi\|_0^2 \\ &= \|\tau\|_0^2 - 2\sum_i n^{-1} U^\varphi(a_i) + \|\varphi\|^2 \quad (U^\varphi \text{ continuous, } \tau \text{ as above}).\end{aligned}$$

4. From now on, we fix $N = 3$ to simplify some formulations. Let F be a surface dividing R^3 in two domains, a bounded one, denoted by D, and an unbounded one, denoted by D_∞. Let $f(x)$ be a real-valued continuous function on F and $u(x, f)$ the *solution of Dirichlet's problem* that satisfies the following conditions: u is continuous in $R^3 \cup \{\infty\}$, harmonic in $D \cup D_\infty$, $u(x, f) = f(x)$ $(x \in F)$, $u(\infty, f) = 0$. We assume all points on F to be *regular* with respect to both D and D_∞, i. e., we assume the existence of such $u(x, f)$ for any continuous f. On F, we use the topology induced from R^3; e.g., ∂F_r in the proof of Lemma 4 is a curve.

THE EXTREMAL POINTS METHOD

5. This method, conceived by Leja ([12], [13] and [14], p. 458), was worked out for the three-dimensional case by Górski [6]. We summarize his results in this section (we replaced f by $-f$ in Górski's formulae in order to set off the parallelism to the plane case).

Take a measure $\mu_n \epsilon T_n$ (2.2) such that

$$(5.1) \qquad I_n \overset{\text{df}}{=} I_0(\mu_n, f) \leqslant I_0(\tau, f) \overset{\text{df}}{=} \frac{n}{n-1}\|\tau\|_0^2 + 2\int f d\tau \qquad (\tau \epsilon T_n).$$

The system of points a_j for $\tau = \mu_n$ will be denoted by

$$(5.2) \qquad \{e_0, \ldots, e_{n-1}\} = \{e_{0,n-1}, \ldots, e_{n-1,n-1}\}$$

and called the $(n-1)$*-th extremal system* of F. It may not be unique; we think, for any n, a concrete extremal system (thus, a concrete μ_n) to be chosen.

Put else

$$(5.3) \qquad I(\mu, f) = \|\mu\|^2 + 2\int f d\mu \qquad (\mu \epsilon \mathscr{E}, \mu \subset F).$$

Take any sequence of indices $n(k) \to \infty$. Then, as proved by Górski, we have

LEMMA 1. *There exists a sequence $k(i)$ and a measure $\mu^* \subset F$ such that*

$$(5.4) \qquad \mu_{n(k(i))} \to \mu^*, \qquad I_{n(k(i))} \uparrow I,$$

where ((2.1), (5.3))

$$I \overset{\text{df}}{=} I(\mu^*, f) \leqslant I(\mu, f) \qquad (\mu \epsilon T).$$

Denoting by b a suitable constant, there exists a set F_0 such that

$$(5.6) \qquad U^{-\mu^*}(x) \to f(x') - b \qquad (x \to x' \epsilon F_0 \subset F).$$

F_0 is always of positive capacity [2], but it may not equal F.

6. We ask now for conditions that ensure $F_0 = F$. A necessary one is of course the continuity of f itself.

Let us now adapt a concept and results of Siciak [17], due to him in the case $N = 2$, for somewhat more general f.

Df$_1$. We call f *solvable* and write $f \epsilon R_1$ if it can be represented as $f(x) = U^{-\varphi}(x) + b$ $(x \epsilon F)$, where φ is a measure in T (2.1) with continuous potential, and b is a constant number.

It is then immediate that (see section 4)

$$(6.1) \qquad u(x, f) = U^{-\varphi}(x) + b \qquad (x \epsilon D).$$

Df$_2$. We call f *solvable* and write $f \epsilon R_2$ if it is the trace on F of a function f^* continuous, subharmonic and bounded in R^3, harmonic in the closed exterior of some ball $B \supset F$ and satisfying the following equation

$$\iint_{\partial B} (\partial f^*/\partial n)\, dS = -4\pi, \tag{6.2}$$

the differentiation being in the exterior normal direction to ∂B and the integration — with respect to the area on ∂B.

The definitions above are equivalent. Indeed, let $f \in R_1$, then $f^* = U^{-\varphi}+b$ satisfies Df$_2$, the integral on the left-hand side of (6.2) being equal to $-4\pi\varphi(F)$ ([8], p. 43). Conversely, if $f \in R_2$, then ([15] and [16]) $f^*(x) = U^{-\psi}(x)+H(x)$ with $\psi \subset B$, $\psi \geqslant 0$ and $H(x)$ harmonic in R^3. Moreover, by (6.2), $\psi(B) = 1$. Being bounded in R^3 (because of $U^{-\psi}(\infty) = 0$), H is constant ([11], p. 282). Now, by the theory of balayage [3] there exists a positive measure $\psi_F \subset F$ with $U^{\psi_F}(x) = U^{\psi}(x)$ on F (all points of F being regular) and $\psi_F(F) = a \leqslant 1$ (see e. g. the argument concerning (17.2) in [9], p. 272). Then by (3.3) we have

$$\varphi = \psi_F + (1-a)\,\eta \in T. \tag{6.3}$$

This measure satisfies Df$_1$, since U^{ψ_F} and U^{η} are continuous by the regularity of F (section 4; see e.g. [3]).

Df$_3$. We call f *solvable* and write $f \in R_3$ if for some sequences $n(k)$, $k(i)$ the measure μ^* from Lemma 1 has a continuous potential and $f(x) = U^{-\mu^*}(x)+b$ $(x \in F,$ b a constant).

If $f \in R_3$, then $f \in R_1$, of course. Conversely, if $f \in R_1$, then for $\mu \in T$

$$I(\mu, f) = \|\mu\|^2 + 2\int (U^{-\varphi}+b)\, d\mu = \|\mu-\varphi\|^2 - \|\varphi\|^2 + 2b. \tag{6.4}$$

Thus the unique measure minimizing it in T is φ. Hence $\mu^* = \varphi$ (see Lemma 1) and Df$_3$ is satisfied.

So $R_1 = R_2 = R_3$ and, from now on, we write simply R instead of R_1, R_2 or R_3. For $f \in R$, $\mu^* = \varphi$ independently of $\{n(k)\}$ — hence the whole sequence μ_n is convergent, viz.,

$$\mu_n \to \varphi \quad (n \to \infty, f \in R). \tag{6.5}$$

So $F_0 = F$ by (5.6), (6.1) and Df$_1$.

The term solvable is originally motivated by a definition related to our Df$_3$, [17]. But a more general motivation may be given: for solvable f, the Dirichlet's problem is solvable by the Gauss's variational method, i.e., by minimizing $\|\mu\|^2 + 2\int f\, d\mu$ for $\mu \in \mathscr{E}$. Nevertheless, the class R_G of Gauss solvable continuous functions is larger than R: one can show by (6.4) that a definition of R_G is obtained from Df$_1$ by dropping the condition $\varphi \in T$ there and setting $b = 0$ (since a $b \neq 0$ can be absorbed by the potential, see (3.3)). So a problem suggests itself:

P 457. *Give a generalization or a variant of the extremal points method for which the related class R_3 is identical with R_G.*

We give now some information concerning the class R.

LEMMA 2. *The class R is convex, i.e., if, for $i = 1, \ldots, n$, $f_i \in R$ and a_i are positive numbers with $\sum a_i = 1$, then $\sum a_i f_i \in R$.*

This is obvious from Df_1.

In particular, for $f \in R$ and $0 \leqslant \lambda \leqslant 1$ we have $\lambda f = \lambda f + (1-\lambda) 0 \in R$, since by Df_1 with $\varphi = \eta$ we have $0 \in R$.

LEMMA 3. *Let F be a C^2-surface and let f_0 satisfy a Lipschitz condition* (7.1). *Then*

1° *for some $\lambda_0 > 0$, $f = \lambda_0 f_0 \in R$,*

2° *for this f, $u(x, f)$ and consequently $U^{\varphi}(x)$ from* (6.1) *satisfies a Lipschitz condition*

$$|U^{\varphi}(x+h) + U^{\varphi}(x)| \leqslant p\,|h| \quad (x \in R^3,\, h \in R^3). \tag{6.6}$$

The last inequality is given in [5], p. 213, for $x \in F$. But $g_h(x) = U^{\varphi}(x+h) - U^{\varphi}(x)$ (h constant) is harmonic when $x, x+h \in D$, and continuous in the closure of this set. Thus the generalization of (6.6) to $x, x+h \in D^-$ is immediate, and so is that to D_{∞}^-, since $g_h(\infty) = 0$. Now, if a function satisfies a Lipschitz condition in D^- and in D_{∞}^-, it does also in $D^- \cup D_{\infty}^- = R^3$. The first part of the lemma is proved by Górski [7]. Both authors admit more general (Liapounoff's) F.

LEMMA 4. *If $F \in C^1$, $\varphi \subset F$ and φ satisfies* (6.6), *then*

$$|\varphi(E)| \leqslant \frac{p}{2\pi} |E| \tag{6.7}$$

for any Borel set $E \subset F$ of area $|E|$.

Remark. We do not suppose $\varphi \in T$ in this lemma.

Proof. Let for any $h > 0$ and any measure μ, μ_h be the h-average, viz., the measure of density

$$\mu_h'(x) \overset{\mathrm{df}}{=} \mu\big(B(x, h)\big)/|B(x, h)|,$$

$B(x, h)$ being the ball of centre x and radius h and $|\ |$ denoting volume. It is well known that $U^{\varphi_h}(x) = \int_{B(x,h)} U^{\varphi}(y)\, dV / |B(x, h)|$, where dV denotes the volume element — this is immediate by Fubini's Theorem ([15], § 6.23, and [18]). So U^{φ} being continuous, $U^{\varphi_h} \in C^1$ and $U^{\varphi_{hh}} \in C^2$, where $\varphi_{hh} = (\varphi_h)_h$. Now, (6.6) implies $|U^{\varphi_{hh}}(x) - U^{\varphi_{hh}}(y)| \leqslant p\,|x-y|$.

Let x_0 be any point of F. Take the coordinate system (t_1, t_2, t_3) with x_0 as its origin and t_3-axis normal to F. There exists an $r^* > 0$ such that for any $r \in (0, r^*\rangle$ the maximal connected part $F_r = F_r(x_0)$ of F, containing x_0 and lying within the cylinder Z: $t_1^2 + t_2^2 < r$, can be represented by $t_3 = g(t_1, t_2) \in C^1$, and the domain E consisting of points in Z with $g(t_1, t_2) - \delta < t_3 < g(t_1, t_2) + \delta$ is disjoint with $F - F_r$ for small $\delta > 0$.

The boundary of E is piecewise smooth. Thus ([8], p. 155-156)

$$|\varphi_{hh}(E)| = \left| \frac{1}{4\pi} \int\limits_{\partial E} \frac{\partial U^{\varphi hh}}{\partial n} dS \right| \leqslant \frac{p}{4\pi} |\partial E|$$

(differentiation in the normal direction, integration with respect to area; $|\partial E|$ means area). But $|\varphi|(\partial E) = |\varphi|(\partial F_r) = 0$, since $\|\varphi\|^2$ is finite and ∂F_r has the capacity null ([9], lemma 8), so $\varphi_{hh}(E) \to \varphi(E)$ $(h \to 0)$ and $|\varphi(F_r)| = |\varphi(E)| \leqslant (p/4\pi)|\partial E|$. Setting $\delta \to 0$ we obtain $|\varphi(F_r)| \leqslant$ $\leqslant (p/2\pi)|F_r|$. Now, $\{F_r(x_0): r \leqslant r^*, x_0 \in F\}$ is a base for the class of Borel sets on F, so the inequality obtained generalizes to (6.7) by a routine argument.

THE RESULT

7. Our aim is to estimate the order of convergence of Leja-Górski method under assumptions of Lemma 3. Then (see the remark following Lemma 2) with any $\lambda \in (0, \lambda_0\rangle$, $\lambda f = f_0 \in R$ and $u(z, f_0) = u(z, f)/\lambda$. Thus it is sufficient to give the order of convergence for a Lipschitzian solvable f.

THEOREM. *Let $F \in C^2$, $f \in R$ and let f satisfy a Lipschitz condition on F:*

$$(7.1) \qquad |f(x_1) - f(x_2)| \leqslant c|x_1 - x_2| \quad (x_1, x_2 \in F;\ c \text{ a constant}).$$

Then (6.5) is valid, $u(x, f) = U^{-\varphi}(x) + b$ $(x \in D)$ and

$$(7.2) \quad U^{-\varphi}(x) = U^{-\mu_n}(x) + g(x), \quad |g(x)| \leqslant M(x) O(n^{-1/6}) \quad (x \in D \cup D_\infty),$$

$M(x)$ being a continuous function of x alone in $D \cup D_\infty$.

If, moreover, $f \in C^2$ on F, we have

$$(7.3) \qquad b = b_n + O(n^{-1/6}), \quad b_n = \|\mu_n\|_0^2 + \int f d\mu_n,$$

where O is the Landau symbol.

For symbols used, see section 4 and section 6 Df_1, (5.1-2), (3.2), and (2.2).

Our thanks are due to Mr. J. Siciak, whose suggestion to use the class R in an earlier paper helped us to eliminate some unessential devices we needed when working immediately with an arbitrary smooth f.

PROOF OF THE THEOREM

8. Energy estimation. Let ψ_{in} be the measure of mass $1/n$ spread uniformly over the sphere ∂B_i, $B_i = \{x: |x - e_i| < n^{-1/2}\}$ $(i = 0, \ldots, n-1)$ (5.2). For $x \in B_i^-$ $(x \notin B_i^-)$, $l(y) = 1/|x - y|$ is superharmonic (harmonic) for $y \in B_i^-$, hence

$$(8.1) \qquad U^{\psi_{in}}(x) = \int l(y) d\psi_{in} \leqslant l(e_i)/n = U^{\varepsilon(e_i)/n}(x) = 1/n|x - e_i|$$

((3.1), (2.2)) with a strong inequality only for $x \in B_i$ (observe the continuity on ∂B_i). Thus

$$\|\psi_{in}\|^2 = n^{-3/2},$$
$$(\psi_{in}, \psi_{kn}) \leqslant \int U^{\varepsilon(e_i)/n} d\psi_{kn} = \int U^{\psi_{kn}} d\varepsilon(e_i)/n \leqslant \big(\varepsilon(e_k)/n, \varepsilon(e_i)/n\big),$$

and for $\psi_n = \sum_i \psi_{in}$ we obtain

$$\|\psi_n\|^2 = \sum_{i,k} (\psi_{in}, \psi_{kn}) = \sum_{i\neq k} + \sum_{i=k} \leqslant \|\mu_n\|_0^2 + n^{-1/2}. \tag{8.2}$$

On the other hand, U^φ satisfies, by Lemma 3, a Lipschitz condition — in particular, $|U^\varphi(x) - U^\varphi(e_i)| \leqslant p|x-e_i|$, thus

$$\big|(\varphi, \psi_{in}) - \big(\varphi, \varepsilon(e_i)\big)/n\big| = \Big|\int \big(U^\varphi(x) - U^\varphi(e_i)\big) d\psi_{in}\Big| \leqslant pn^{-1/2}n^{-1}. \tag{8.3}$$

Applying (6.4) to generalize $I(\mu, f)$ to $\mu \notin F$, we obtain by (8.2-3) and by Lemma 1

$$\begin{aligned} I(\psi_n, f) &= \|\psi_n\|^2 + 2\int (U^{-\varphi} + b) d\psi_n \\ &\leqslant \|\mu_n\|_0^2 + 2\int (U^{-\varphi}+b) d\mu_n + (1+p)n^{-1/2} = I_n + \ldots \leqslant I + (1+p)n^{-1/2}, \end{aligned} \tag{8.4}$$

so by (6.4), Lemma 1 and by $\mu^* = \varphi$,

$$\|\psi_n - \varphi\|^2 = O(n^{-1/2}). \tag{8.5}$$

9. Heuristical preparation. The ultimate move is in applying theorem VI from [9] or its simpler version (theorem VII) which asserts that $[\sigma]^3 \leqslant c\|\sigma\|^2$, where c depends on the bound for density of $\sigma \subset F$ and $[\ ]$ is a specific norm such that $|U^\sigma(x)| \leqslant [\sigma]M(x)$ $(x \notin F)$. So $[\psi_n - \varphi] = O(n^{-1/6})$ would follow from (8.5). But $\psi_n - \varphi$ neither lies on F nor any surrogate of its density with respect to F is known to be bounded. Hence, we cannot use the theorem mentioned, but rather its proof with suitable changes.

We will follow the lines of section 17 and a part of section 13 in [9]. *Some points explained there will be only sketched here.* To avoid geometrical complications (in proving that $(S - \delta CS) \in L$ for $S \in L$; notation of [9]) we change to another norm (11.2), somewhat differing from $[\ ]$ but giving the same effect on potentials.

10. Regular parameters. Take a partition $F = \bigcup_{j=1}^{q} F_j$ (with $\overline{F}_j \cap \overline{F}_k \subset \partial F_j \cap \partial F_k$ for $i \neq k$), where any F_j can be represented by a "regular parametrization":

$$F_j\colon\ x = \varphi_j(u, v) \qquad ([u, v] \in Q = \langle 0, 1\rangle \times \langle 0, 1\rangle) \tag{10.1}$$

with the following properties: 1° $\varphi_j \in C^2$ in some open domain containing Q and maps it into F, 2° there exists a number $\vartheta > 0$ such that

$$\varrho\big(A_j v_0, A_j(v_0+h)\big) \geqslant \vartheta h \quad (0 \leqslant v_0 < v_0+h \leqslant 1), \tag{10.2}$$

where, for any fixed v, $A_j v$ denotes the arc $x = \varphi_j(u, v)$ $(u \in \langle 0, 1\rangle)$, 3° the same holds when the parts of the variables u, v are interchanged. Together with the boundedness of derivatives, this implies

$$\theta_1 \leqslant |\varphi_j(A)|/|A| \leqslant \theta_2, \quad \theta_1 \leqslant |\varphi_j(S)|/|S| \leqslant \theta_2 \quad (j = 1, \ldots, n), \tag{10.3}$$

with some positive constants θ_1, θ_2, for any smooth arc $A \subset Q$ (bars denote length) and any domain $S \subset Q$ with piecewise smooth boundary (bars denote area).

The possibility of the construction above follows by triangulating regularly F and dividing each "triangle" in 4 "quadrilaterals" with C^2-sides, so small as to project on a tangent plane under small angles; then a one-to-one C^2-mapping of this projection onto Q enables us to define φ_j with the required properties.

11. The norm [ ]*. Put $F^* = F \cup \bigcup_{i=0}^{n-1} \partial B_i$ (section 8) and, generally, for any set $E \subset F$ let E^* consist of points of E and of ∂B_i's projecting themselves onto E:

$$E^* = \{x \in F^*:\ \varrho(x, F) = |x - x'|,\ x' \in E\}.$$

Observe that if $n^{-1/2} \leqslant r_0 =$ minimal radius of curvature of F, then to any $x \in F^*$ there corresponds only one projection $x' \in F$. We limit ourselves to such values of n. For any $S \subset Q$, abbreviate

$$S_j = \varphi_j(S). \tag{11.1}$$

Put now, for $\sigma \subset F^*$,

$$[\sigma]^* = \sup_j \sup_{S \in \Lambda} |\sigma(S_j^*)|, \tag{11.2}$$

where Λ denotes the class of all axial rectangles contained in Q. Lemma 7 of [9] holds with $[\sigma]^*$ instead of $[\sigma]$, since we used essentially [ ]* in its proof:

$$|U^\sigma(x)| \leqslant [\sigma]^* M(x) \quad (x \notin F^*,\ \sigma \subset F,\ M(x) \text{ continuous}). \tag{11.3}$$

12. We assert that

$$[\mu_n - \varphi] = O(n^{-1/6}). \tag{12.1}$$

Assume, on the contrary, that there exist sequences n_k and i_k, $S^k \in \Lambda$, such that $n_k \to \infty$ and

$$\pm(\varphi - \mu_{n_k})(S_{i_k}^k) \geqslant c_k^* \quad (c_k^* n_k^{1/6} \to \infty\ (k \to \infty)). \tag{12.2}$$

We can assume that the sign in (12.2) is always $+$. Indeed, let for a k the minus sign occur. Extend the sides of S^k to divide Q in 9 rectangles. Denote their images under φ_{i_k} by $F_{q+1}, \ldots, F_{q+8}, F_{q+9} = S^k_{i_k}$. The common boundary points of any F_i's are to be counted to (any) one of them only. The entire mass being null, $\sum_{i=1}^{q+8} (\varphi - \mu_n)(F_i) \geqslant c_k$, so for an $i = i'$ we have $(\varphi - \mu_n)(F_{i'}) \geqslant c_k^*/(q+8)$. Thus, taking $F_{i'}$ for $S^k_{i_k}$ and diminishing $(q+8)$ times c_k^*, we have (12.2) with plus sign. With this new c_k^*, the first minus in (12.2) can be dropped.

Moreover, the range of i_k being $\{1, \ldots, q\}$, we can assume (by passing to a subsequence) — to simplify the formulae — that $i_k = j$ ($k = 1, 2, \ldots$).

Cut a boundary strip $\Delta^k = S^k \cap \delta\partial S^k$ where $\delta = n_k^{-1/6}$; see section 2. We have (radius of B_i) $= n_k^{-1/2} \leqslant \vartheta\delta$ (see (10.2)), provided n_k is sufficiently great, thus

$$(12.3)\qquad \begin{gathered} -\mu_{n_k}(S^k_j) \leqslant -\psi_{n_k}\big((S^k - \Delta^k)^*_j\big), \\ \varphi(S^k_j) - p^* n_k^{-1/6} \leqslant \varphi\big((S^k - \Delta^k)_j\big) \qquad (p^* = \theta_2 \cdot (p/2\pi) \cdot 4) \end{gathered}$$

(Lemma 4 and (10.3)). Put

$$(12.4)\qquad \sigma_n = \varphi - \psi_n.$$

By the foregoing inequalities, we have

$$(12.5)\qquad \sigma_{n_k}\big((S^k - \Delta^k)^*_j\big) \geqslant c_k, \qquad c_k n_k^{1/6} = c_k^* n_k^{1/6} - p^* \to \infty \qquad (k \to \infty).$$

But $S^k - \Delta^k \in \Lambda$, so for any k we have

$$(12.6)\qquad m_k \overset{\mathrm{df}}{=} \sup_i \sup_{S \in \Lambda} \sigma_{n_k}\big((S_i)^*\big) \geqslant c_k.$$

As easily seen, $\sigma_{n_k}\big((S_i)^*\big)$ is a continuous function of $S \in \Lambda$, provided n_k is great enough. So there exists a $C^k \in \Lambda$ and an i (depending on k) with $\sigma_{n_k}\big((C^k_i)^*\big) = m_k$.

Now, let Δ^k denote a boundary strip in C^k:

$$(12.7)\qquad \Delta^k = C^k \cap \delta\partial C^k, \qquad \delta = n_k^{-1/6},$$

and put $A = \Delta_i^{k*}$. Observe that $\sigma_{n_k}(A) \geqslant 0$ — otherwise $\sigma_{n_k}\big((C^k - \Delta^k)^*_i\big) > m_k$ while $(C^k - \Delta^k) \in \Lambda$ — hence (see (12.4) and (6.7))

$$\psi_{n_k}(A) \leqslant \varphi(A) = \varphi(\Delta^k_i) \leqslant p^* \cdot 4 n_k^{-1/6}.$$

Thus, for great k,

$$(12.8)\qquad |\sigma_{n_k}|\big((C^k - \Delta^k)^*_i\big) \geqslant \tfrac{1}{2} m_k.$$

13. In what follows we will denote some sets which, in our present considerations, play the role of S_+, $K(m, \nu)$, E, A from [9], sections 14, 17 (as to A, see (12.7)) and the measures replacing ν, α, by the same symbols. Thus, the symbols mentioned have a new meaning from now on.

We put

$$\nu = |\sigma_{n_k}| = \varphi + \psi_{n_k}, \quad S_+ = C_i^{k^*}. \tag{13.1}$$

Let K be now the following class of measures on F^*:

$$K = K(m_k, \nu) = \{\sigma : \sigma \subset F^*, |\sigma| \leqslant \nu, \sigma(F^*) = 0, \sigma(S_+) = m_k\}. \tag{13.2}$$

ν has a continuous potential (since ψ_n and φ have). $K(m_k, \nu)$ is compact (by [1], III. 2, see [9], section 14) and $\|\sigma\|^2$ continuous in it ([9], Lemma 2), so a minimal measure $\alpha \in K$ exists with $\|\alpha\|^2 \leqslant \|\sigma\|^2$ $(\sigma \in K)$, in particular

$$\|\sigma_{n_k}\|^2 \geqslant \|\alpha\|^2. \tag{13.3}$$

We show, as in the proof of theorem II [9] that we may suppose $\alpha \geqslant 0$ on S_+.

Form the set $E = \{x : \varrho(x, F) = |x - x'| < 2n_k^{-1/6},\ x' \in S_+ \cap F\}$. Put $P = \partial E$. Then P separates S_+ from $F^* - S_+$ (provided n is great enough, for the argument see [9], section 17). Moreover, with ϑ from (10.2) we have

$$\varrho(S_+ - A, P) \geqslant \delta' \overset{\mathrm{df}}{=} \tfrac{1}{2}\vartheta n_k^{-1/6}, \tag{13.4}$$

provided n_k is so great as to make F sufficiently flat in the $2\delta'$-neighbourhood of any point of F. The elementary proof of this fact is based on the observation that by (12.7) and (10.2) we have $\varrho\big((S_+ \cap F) - A, \partial(S_+ \cap F)\big) \geqslant 2\delta'$; we omit the details.

Clearly, by the definition of C_i^k (after (12.6)) and by (12.8), (13.1-2),

$$\alpha(A) \leqslant \nu(A) \leqslant \tfrac{1}{2} m_k. \tag{13.5}$$

Take a partition

$$S_+ \cap F - A = \bigcup_{i=1}^{q} \mathscr{F}_i,$$

where $\mathscr{F}_i$ are mutually disjoint and each $\mathscr{F}_i$ is contained in a ball $\mathscr{B}_i = \{x : |x - x_i| < \frac{1}{4}\delta'\}$ $(x_i \in \mathscr{F}_i)$. So for $\mathscr{B}_i^* = \{x : |x - x_i| > \delta'\}$ we have $\partial B_i \subset E^-$ (since $\vartheta \leqslant 1$). We can (and do) suppose that $q \leqslant c^*(\delta')^{-2} = cn^{1/3}$ (see [9], § 17). Put (see section 2)

$$\alpha_0 = \alpha \mid A, \quad \alpha_i = \alpha \mid \mathscr{F}_i \quad (i = 1, \ldots, q).$$

Introducing the measures β_i, obtained by balaying α_i onto $P \cup F^* - E$ as in [9] (17.2-4) and repeating the subsequent considerations, we obtain

$$\|\alpha\|^2 \geqslant \frac{3}{4 \cdot \frac{1}{4}\delta'} q \left(\frac{m_k}{2q}\right)^2 \geqslant c' n_k^{-1/6} m_k^2.$$

By (13.3), (12.4), (8.5) and (12.6), this implies $O(n_k^{-1/2}) \geqslant c' n_k^{-1/6}(c_k)^2$, so $n_k^{1/2} O(n_k^{-1/2}) \geqslant c'(c_k n_k^{1/6})^2 \to \infty$ by (12.5), which gives the desired contradiction. So (12.1) is proved and (7.2) follows from (11.3).

14. If we want to calculate b, we may integrate the equality in Df_1 (section 6) with respect to φ, thus obtaining

$$b = \|\varphi\|^2 + \int f d\varphi = I - \int f d\varphi .$$

The approximate formula will be (see (5.1))

$$b \approx b_n = \|\mu_n\|_0^2 + \int f d\mu_n = I_n - \int f d\mu_n .$$

Since (Lemma 1 with $\mu^* = \varphi$, (6.5)) $I_n \leqslant I \leqslant I(\mu, f)$ for all μ of finite energy (taking (6.4) as the definition), in particular, for $\mu = \psi_n$, we have, by (8.4), $I \leqslant I_n + (1+p)n^{-1/2} \leqslant I + (1+p)n^{-1/2}$. But in order to estimate $\int f d\mu_n - \int f d\varphi$, we need second derivatives of f (with respect to local coordinates on F). Applying then the proof of Lemma 7 [9] (simply substituting there our f, expressed as a function of u, v, for $f^i(u, v)$), we obtain

$$\left| \int f d(\mu_n - \varphi) \right| \leqslant [\mu_n - \varphi]^* M ,$$

where M depends on f only. Thus, by (12.1), $b = b_n + O(n^{-1/6})$. This completes the proof of our theorem.

15. The derivatives may be obtained by differentiation of the integrand, and we have the convergence estimation (see (6.1))

$$\begin{aligned} |\operatorname{grad} u(x, f) - \operatorname{grad} U^{-\mu_n}(x)| &\leqslant [\mu_n - \varphi]^* M_1(x) \\ &= O(n^{-1/6}) M_1(x) \qquad (x \in D \cup D_\infty), \end{aligned}$$

where $M_1(x)$ is a continuous function. The proof runs in full analogy to that of Lemma 7 [9]. Similar results hold for higher derivatives. Observe that in questions concerning derivatives — thus in many applications — the constant b disappears, so the additional assumption $f \in C^2$ is not necessary there.

GENERALIZATION TO HIGHER DIMENSIONS

16. The extremal points method will do in the N-dimensional space ($N \geqslant 4$) as well. All basic expressions are given in sections 2-3 just for general N. Górski's results [6, 7] generalize with his own proofs, and the whole section 6 is valid — with the obvious change from 2- and 3-dimensional sets to the $N-1$ and N-dimensional ones, in this and subsequent sections. In sections 7-15 (in particular, in the theorem) the class C^2 is to be replaced by C^{N-1}, and $n^{-1/6}$ by $n^{-1/3(N-1)}$. The balls B_i (section 8) change now their radii to $n^{-1/(N-1)}$. Additional hints and some explanation may be found in [9], section 20.

17. Thus, it may seem that the convergence of the extremal points method becomes the worse the higher is the dimension N. But we have to remember that with increasing N all numerical methods become necessarily more elaborate, since to cover, say, F by a sequence of ε_k-nets, where $\varepsilon_k = O(k^{-1})$, we need $n_k = O(k^{N-1})$ points for the k-th net. In particular, if an extremal points system has to be such a net (in the average sense), it has to consist of $n_k = O(k^{N-1})$ points. When referred to k, the order of convergence of the extremal points method is thus $O(k^{-1/2}\log k)$ on the plane [10], $O(k^{-1/3})$ in R^N $(N \geqslant 3)$.

REFERENCES

[1] N. Bourbaki, *Intégration*, Paris 1952.

[2] H. Cartan, *Théorie du potentiel Newtonien*, Bulletin de la Société Mathématique de France 73 (1945), p. 74-106.

[3] — *Théorie générale du balayage en potentiel Newtonien*, Annales de l'Université de Grenoble 22 (1946), p. 221-280.

[4] O. Frostman, *Potentiel d'équilibre et capacité des ensembles*, Commentaires du Séminaire Mathématique de l'Université de Lund 3 (1935), p. 118.

[5] N. M. Giunter, *Potential theory*, Warszawa 1957 (in Polish; Russian original, Moscow 1953).

[6] J. Górski, *Méthode des points extrémaux de résolution du problème de Dirichlet dans l'espace*, Annales Polonici Mathematici 1 (1955), p. 418-429.

[7] — *Une remarque sur la méthode des points extrémaux de F. Leja*, ibidem 7 (1959), p. 62-69.

[8] O. D. Kellog, *Foundations of potential theory*, Berlin 1929.

[9] W. Kleiner, *On the equilibrium of signed measures*, Colloquium Mathematicum 11 (1964), p. 257-278.

[10] — *Sur les approximations de M. Leja dans le problème plan de Dirichlet*, Annales Polonici Mathematici 15 (1964), to appear.

[11] M. Krzyżański, *Partial differential equations of second order*, part I, Warszawa 1957 (in Polish; English edition in preparation).

[12] F. Leja, *Sur une famille de fonctions harmoniques dans le plan, liées à une fonction donnée sur la frontière d'un domaine*, Bulletin International de l'Académie Polonaise des Sciences et des Lettres, 1936, p. 79-91.

[13] — *Une méthode élémentaire de résolution du problème de Dirichlet dans le plan*, Annales de la Société Polonaise de Mathématiques 23 (1950), p. 230-245.

[14] — *The theory of analytic functions*, Warszawa 1957 (in Polish).

[15] T. Radó, *Subharmonic functions*, New York 1949.

[16] F. Riesz, *Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel, II*, Acta Mathematica 54 (1930), p. 321-360.

[17] J. Siciak, *Some applications of the method of extreme points*, Colloquium Mathematicum 11 (1964) p. 209-250.

[18] J. M. Thompson, *Distribution of mass for averages of Newtonian potential functions*, Bulletin of the American Mathematical Society 41 (1935), p. 744-752.

JAGELLONIAN UNIVERSITY

Reçu par la Rédaction le 16. 9. 1963