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ALMOST PERIODIC EXTENSIONS OF FUNCTIONS

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1. Mycielski [10], answering a question of Marczewski and Ryll-Nardzewski, proved that there exist sequences of positive numbers $\{t_n\}$ such that for every sequence $\varepsilon_n = 0, 1$ there is a continuous periodic function f taking values ε_n at the points t_n respectively. Lipiński [8] exhibited a sequence $\{t_n\}$ such that every bounded real function on it can be extended to a continuous periodic function. In both results it is essential and sufficient that t_n's increase rapidly enough, e.g. in the theorem of Mycielski at least as rapidly as $(3+\alpha)^n$, $\alpha > 0$. This result was strengthened by Ryll-Nardzewski who showed that for $t_n = 3^n$ a continuous periodic function with $f(t_n) = \varepsilon_n$ still can be found for any sequence $\varepsilon_n = 0, 1$. On the other hand, no sequence $t_n = O(2^n)$ has this property [12].

Here we are concerned with interpolation by means of a continuous almost periodic function (Bohr function, in the sequel denoted by a. p.). More exactly, we are interested in two following properties of a subset A of the real axis L:

$\boldsymbol{I}$: A has property $\boldsymbol{I}$ (or is an $\boldsymbol{I}$-set or belongs to the class $\boldsymbol{I}$: $A \in \boldsymbol{I}$) if every bounded, real or complex valued function on A which is uniformly continuous on A with respect to the usual metric of L can be extended to an a. p. function over L.

$\boldsymbol{I}_0$: Terms and definition are analogous but the uniform continuity is not assumed.

Obviously $\boldsymbol{I}_0 \to \boldsymbol{I}$ and the equivalence $\boldsymbol{I}_0 \equiv \boldsymbol{I}$ holds if $\sigma(A) > 0$, where $\sigma(A) = \inf\{|x-y| : x \neq y;\ x, y \in A\}$. Property $\boldsymbol{I}_0$ will be applied also to other groups than L: if G is an Abelian topological group, then the set $A \subset G$ has property $\boldsymbol{I}_0$ (is an $\boldsymbol{I}_0$-set, $A \in \boldsymbol{I}_0$), if every bounded complex valued function on A can be extended to a *continuous* a. p. function on G.

Hartman [5] proved that no sequence $\{n_k\}$ (k integer) is an $\boldsymbol{I}$-set.

The same can be stated about any sequence n^a $(a > 0)$, because if a is not an integer then $\{n^a\}$ is *equidistributed* on L [2], i.e. we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n^a) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\,dt$$

for every a. p. function; hence f is uniquely determined by values it takes at the points n^a, even if one such point, n_0^a say, is omitted. Thus no a. p. function satisfies the conditions $f(n^a) = 0$ for $n \neq n_0$ and $f(n_0) = 1$.

We do not restrict our attention to sequences of points. We rather ask about $\boldsymbol{I}$-sets of the form $\bigcup_n I_n$, where I_n are (non-degenerated) intervals. We intend to show that such sets do exist and we shall describe a class of them. The main tool in our reasonings will be the Bohr compactification of L. We shall use also Bohr compactification of other Abelian locally compact groups. We remind: if G is such a group, we get the needed compactification $\tilde{G}$ by passing to the dual group $\hat{G}$, neglecting its topology, thus obtaining a discrete group $\hat{G}_d$, and taking the dual of $\hat{G}_d$. Another method to obtain $\tilde{G}$ is the mapping of G by means of the continuous homomorphism $\varphi(t) = \{\chi(t)\}$ $(\chi \in G)$ into the thorus of dimension card$(\hat{G})$ and taking the closure of $\varphi(G)$ in it. If we understand by an a. p. function on G a *continuous* a. p. function with respect to the assumed topology, then, by the main approximation theorem, every a. p. function, considered as a function on $\varphi(G)$, can be extended to a continuous function over $\tilde{G}$ and vice versa: if f is continuous on $\tilde{G}$, then the function $f(\varphi(t))$ is a. p. on G.

For any set $A \subset G$, $\tilde{A}$ will denote the closure of $\varphi(A)$ in $\tilde{G}$. We are chiefly interested in the case $G = L$ and we shall use the notation $\tilde{L} = K$. Obviously, for $\chi \in \tilde{L}$, $\chi(t) = e^{i\lambda t}$ for some real λ and φ is a monomorphism. So, in the sequel, for any $A \subset L$, $\tilde{A}$ denotes the closure of A in the topology of K (the "weak" closure), whereas $\bar{A}$ means the usual closure in L (note that $\bar{A} \subset \tilde{A}$).

The first example of an application of K may be the immediate proof of the following proposition:

If $a > 0$ is not an integer, then no non-constant 0-1-*sequence can be identic with the sequence $f(n^a)$ for any a. p. function.*

In fact, the sequence $\{n^a\}$ is equidistributed and thus dense in K. The group K being connected, no 0-1-function can be continuous on its dense subset unless it is constant.

Another application of the Bohr compactification method will appear in the proof of the next theorem, which is not a new result, since it was stated (without proof) e. g. in [10]:

THEOREM 1. *For a subset E of an Abelian topological group G to be an $\boldsymbol{I}_0$-set it is sufficient that every function on E taking values* 0 *and* 1 *can be extended to a continuous a. p. function over G.*

Proof. The assumed property is equivalent to the fact that every 0-1-function on E can be extended to a continuous function over $\tilde{E}$. But then it still must be true for every function on E assuming only a finite number of values because of its being a linear aggregate of 0-1-functions. Now, if a bounded function $h(t)$ on E is given, we represent it as a uniform limit of a sequence $\{h_n\}$ of functions with finite range. Since every h_n can be extended over $\tilde{E}$, the same holds for their limit function h. It remains to extend h continuously from $\tilde{E}$ to the whole group $\tilde{G}$.

It is obvious that in the above theorem the values 0 and 1 can be replaced by any two distinct complex numbers.

Using Theorem 1 we immediately see that a set E is an $\boldsymbol{I}_0$-set if and only if it has the following property:

($\mathbf{S}_0$) If E is split arbitrarily into two parts, then these parts have in $\tilde{G}$ no cluster point in common.

It is also easily seen that the closure in G of an infinite $\boldsymbol{I}_0$-sequence in G is homeomorphic to the Čech-Stone compactification $\beta(N)$ of the set N of positive integers. Thus the existence of sequences in L which are $\boldsymbol{I}_0$-sets implies that K contains topologically the space $\beta(N)$.

The existence of $\boldsymbol{I}$-sets which are unions of infinitely many intervals of length $> l > 0$ is an obvious consequence of the results in [10] and [8], quoted above, and of the following

THEOREM 2. *If $A \subset L$ and $l > 0$ are such that every subset $Z \subset A$ with $\sigma(Z) \geqslant l$ is an $\boldsymbol{I}$-set, then the whole set A is an $\boldsymbol{I}$-set.*

The proof will be based on four lemmas.

LEMMA 1. *If f is a bounded real function defined on a subset E of a normal topological space X and the closures of the sets $\{x \in E: f(x) < a\}$ and $\{x \in E: f(x) > b\}$ are disjoint for any numbers a and $b > a$, then f has an extension to a continuous function on X.*

The proof can be omitted.

LEMMA 2. *A set $Z \subset L$ is an $\boldsymbol{I}$-set, if and only if it has the following property:*

($\mathbf{S}$) *if* $S, T \subset Z$ *and* $\operatorname{dist}(S, T) > 0$, *then* $\tilde{S} \cap \tilde{T} = \emptyset$.

The Lemma follows from Lemma 1 by putting $X = K$.

LEMMA 3. *For every $a \in K$ and any numbers $l, \varepsilon > 0$ there exists in K a neighbourhood $U = U(a, l, \varepsilon)$ of a such that if $x, y \in L \cap U$ then either $|x-y| > l$ or $|x-y| < \varepsilon$.*

This follows from the fact that the topology of L and that of K coincide on segments $[a, b]$ the latter being compact.

LEMMA 4. *If* $a \in \tilde{X} \setminus \overline{X}$, *where* $X \subset L$, *then for every* $l > 0$ *there exists a set* $T \subset X$ *such that* $\sigma(T) \geqslant l$ *and* $a \in \tilde{T}$.

Proof. We put $I_n = [nl, (n+1)l]$, $P_n = X \cap I_{2n}$, $Q_n = X \cap I_{2n+1}$,

$$P = \bigcup_n P_n, \quad Q = \bigcup_n Q_n. \tag{1}$$

Obviously we have $X = P \cup Q$, hence $a \in \tilde{P} \cup \tilde{Q}$. Suppose that $a \in \tilde{P}$ (otherwise the proof would be analogous). Let us introduce the sets

$$F_n^k = \overline{U(a, l, 1/k) \cap P_n}.$$

We obviously may assume that $U(a, l, 1/(k+1)) \subset U(a, l, 1/k)$. The diameter of F_n^k is $\leqslant 1/k$ and we have $F_n^k \supset F_n^{k+1}$.

We now define the sets Z_n as follows: if $F_n^1 = \emptyset$, then $Z_n = \overline{P}_n$; if $r(n) > 1$ is the first index k such that $F_n^k = \emptyset$, then $Z_n = \bigcap_{k<r(n)} F_n^k$, and if all sets F_n^k are non-empty, then $Z_n = \bigcap_{k<\infty} F_n^k$; in the last case we put $r(n) = \infty$.

Let N_0 be the set of those n's for which $P_n \neq \emptyset$. For each $n \in N_0$ we select a point $\tau_n \in Z_n$. We will show that a is a cluster point of the set $\{\tau_n: n \in N_0\}$. In fact, if V is an arbitrary weak neighbourhood of the point a we choose another neighbourhood V' and a $k > 0$ such that

$$V' + s \subset V \quad \text{for} \quad -\frac{1}{k} < s < \frac{1}{k}. \tag{2}$$

In view of $a \in \tilde{P}$ and formula (1) there exists a real number t and an integer n_0 such that

$$t \in V' \cap U\left(a, l, \frac{1}{k+1}\right) \cap P_{n_0}.$$

Then $r(n_0) > k+1$ and, consequently, $|t - \tau_{n_0}| \leqslant 1/(k+1)$, since $t, \tau_{n_0} \in F_{n_0}^{k+1}$, and finally in view of (2) we obtain $\tau_{n_0} \in V$. Since $\tau_{n_0} \in \overline{P}_{n_0} \subset \overline{X}$, we have $\tau_{n_0} \neq a$ by the assumption $a \in \tilde{X} \setminus \overline{X}$.

The points τ_n belong to $\overline{P}_n$ by definition, but they need not belong to P_n. Therefore we replace points τ_n by $t_n \in P_n$ in such a way that $|t_n - \tau_n| < \delta/n$, where δ is the distance of a from P or $\delta = 1$ according as to a is a real number or not. Then $a \in \tilde{T}$, where $T = \{t_n: n \in N_0\}$. Since by definition of the sets P_n we have $\sigma(T) \geqslant l$, the lemma is proved.

Now we proceed to the proof of Theorem 2. If the assertion would not hold, then by Lemma 2 there would exist two subsets $P, Q \subset A$ such that $\operatorname{dist}(P, Q) = \varepsilon > 0$ and $\tilde{P} \cap \tilde{Q} \neq \emptyset$. If $a \in \tilde{P} \cap \tilde{Q}$, we put

$$P_1 = P \cap U(a, l, \varepsilon), \quad Q_1 = Q \cap U(a, l, \varepsilon).$$

On account of Lemma 4, there are two subsets $T \subset P_1$ and $S \subset Q_1$ such that $\sigma(T) \geqslant l$, $\sigma(S) \geqslant l$ and

(3) $$a \in \tilde{T} \frown \tilde{S}.$$

Then $\sigma(T \smile S) \geqslant l$ and in view of our assumption we get $T \smile S \in \boldsymbol{I}$. But this leads to a contradiction, since (3) shows that $\tilde{T} \frown \tilde{S} \neq \emptyset$ and so, by Lemma 2, $T \smile S \notin \boldsymbol{I}$.

We finish this section by the remark that no set consisting of intervals I_n of *unbounded* length is an $\boldsymbol{I}$-set. It follows from the fact that the mean value

$$\lim_{T\to\infty} \frac{1}{2T} \int_{x-T}^{x+T} f(t)\,dt$$

of an a. p. function is reached uniformly in x and therefore, if $|I_n| \to \infty$, there is no a. p. function such that

$$\frac{1}{|I_n|} \int_{I_n} f(t)\,dt = 0 \text{ or } 1$$

according as to n is even or odd. It is also easy to prove that a function which is linear in each I_n and reaches from a constant m to a constant $M > m$ cannot be extended to an a. p. function.

One negative result more is worth mentioning: there is no sequence of reals $\{t_n\}$ such that for any bounded sequence $\{\varepsilon_n\}$ a trigonometric polynomial (with real exponents) could be found, taking values ε_n at the points t_n respectively. We omit the proof.

2. Let us denote by I_0 the unit interval $[0, 1]$ and by I its arbitrary translation: $I = a + I_0$, where a is any element of K.

We now remind that a function $f(t)$ defined on L is called *almost periodic in the sense of Stepanoff* (S-a. p.), if it is the limit of a sequence of trigonometric polynomials $\sum_{n=1}^{N} a_n e^{i\lambda_n t}$ in the S-norm

$$\|f\|_S = \sup_t \int_t^{t+1} |f(s)|\,ds.$$

Let $A \subset L$ be an $\boldsymbol{I}$-set. We shall consider only the case that A is the union of disjoint unit intervals (the existence of $\boldsymbol{I}$-sets of this kind was stated in section 1). Hence A can be represented in the form

(1) $$A = E + I_0,$$

where E is the set of the left end-points of the components of A. Consequently we have

(2) $$\tilde{A} = \tilde{E} + I_0$$

(remember that I_0 is a compact set!). Further for a function f defined on the set A only we put

$$f^*(t) = \begin{cases} f(t) & \text{if} \quad t \in A, \\ 0 & \text{if} \quad t \in L \setminus A. \end{cases}$$

THEOREM 3. *Under conditions imposed above on the set A a measurable function f defined on A can be extended to an S-a. p. function if and only if $\|f^*\|_S < \infty$ and f^* is S-continuous i. e. for every $\varepsilon > 0$ we have $\|f_h^* - f^*\|_S < \varepsilon$ if $|h| < \delta(\varepsilon)$, where $f_h(t) = f(h+t)$.*

Proof. The necessity follows from the fact that the conditions imposed on f^* are necessary for a function to be S-a. p. The sufficiency is implied by Lemmas 5-9 given below.

LEMMA 5. *If f^* is S-continuous and $\|f^*\|_S < \infty$, then for every $\varepsilon > 0$ there exists a function g on A, bounded and uniformly continuous, such that $\|f^* - g^*\|_S < \varepsilon$ (the converse implication being also true).*

Proof. Let us put

$$g^h(t) = \frac{1}{h} \int_0^h f^*(t+s)\,ds.$$

Then for all sufficiently small $h > 0$ the restricted function $g = g^h|_A$ satisfies all requirements of the Lemma.

LEMMA 6. *If Z is a union of an arbitrary family of unit intervals I_ξ on the line, then for every $\varepsilon > 0$ and every unit interval I there exist two intervals I_{ξ_1} and I_{ξ_2} such that*

$$|Z \cap I \setminus (I_{\xi_1} \cup I_{\xi_2})| < \varepsilon,$$

where $|\cdot|$ denotes the linear Lebesgue measure.

The proof is obvious.

LEMMA 7. *If F is a closed subset of K and f is a continuous function on F, then*

$$\sup_I \int_{I \cap F} |f(a)|\,da \geqslant \inf_g \max_I \int_I |g(a)|\,da, \tag{3}$$

where I runs over all unit intervals of the form $[\beta, \beta+1]$ ($\beta \in K$) and g over the class $C(f)$ of all continuous extensions of f over K, the differential da referring to the ordinary Lebesgue measure shifted from $[0,1]$ to $[\beta, \beta+1]$.

(Actually the converse implication obviously holds).

Proof. Clearly it is enough to consider non-negative functions f, g only. We assume

$$\max_I \int_I g(a)\,da \geqslant c \quad \text{for every} \quad g \in C(f)$$

and we put

$$Z(g) = \Big\{\beta\colon \int_I g(\alpha)d\alpha \geqslant c,\ I = [\beta, \beta+1],\ g \in C(f)\Big\}.$$

Obviously these sets are closed and non-empty. Moreover, they form a centered family. In fact, we have

$$Z(g_1) \cap \ldots \cap Z(g_n) \supset Z\big(\min(g_1, \ldots, g_n)\big),$$

but $\min(g_1, \ldots, g_n) \in C(f)$. In view of the compactness of K there is a point β_0 common to all $Z(g)$, i. e. we have

$$\int_0^1 g(\beta_0 + t)\,dt \geqslant c \quad \text{for every} \quad g \in C(f).$$

This implies

$$\int_{[\beta_0, \beta_0+1] \cap F} f(\alpha)\,d\alpha \geqslant c, \tag{4}$$

since for every $\varepsilon > 0$ we can find a $g \in C(f)$ such that

$$\int_0^1 g(\beta_0 + t)\,dt - \int_{[\beta_0, \beta_0+1] \cap F} f(\alpha)\,d\alpha < \varepsilon.$$

LEMMA 8. *If A satisfies the assumption of Theorem* 3 *and f is bounded and uniformly continuous on A, then for every $\varepsilon > 0$ there exists an extension of f to an a. p. function g such that*

$$\|g\|_S \leqslant 2\|f^*\|_S + \varepsilon. \tag{5}$$

In other words, almost periodic extensions of f not only exist (which is the sense of property ***I***) but they can be so chosen as to have a "reasonably" small S-norm.

Proof. The set A being an ***I***-set we can extend f (uniquely) to a continuous function f over $\tilde{A}$. Then we have for $\tilde{e} \in L \cap \tilde{E}$

$$\int_{\tilde{e}+I_0} |\tilde{f}(t)|\,dt \leqslant \sup_{e \in E} \int_{e+I_0} |f(t)|\,dt \leqslant \|f^*\|_S.$$

Since $\tilde{A} = \tilde{E} + I_0$ the set $L \cap \tilde{A}$ is the union of unit intervals $\tilde{e} + I_0$, where $\tilde{e} \in L \cap \tilde{E}$. Hence, by Lemma 6, we have for every unit interval $I \subset L$

$$\int_{I \cap \tilde{A}} |\tilde{f}(t)|\,dt \leqslant 2\|f^*\|_S. \tag{6}$$

Now from Lemma 7 we infer that for every $\varepsilon > 0$ there is a continuous function g on K such that $g/_A = \tilde{f}$ and

$$\sup_t \int_t^{t+1} |g(t)|\,dt \leqslant \sup_I \int_{I \cap \tilde{A}} |\tilde{f}(t)|\,dt + \varepsilon.$$

This implies (5) in view of (6).

Now we are able to present the proof of Theorem 3. In view of Lemma 5 the function f can be expanded into a series $f = \sum_{n=1}^{\infty} g_n$, where every g_n is a bounded and uniformly continuous function on A and the series is not only convergent in the S-norm (more exactly we must consider the relation $f^* = \sum_{n=1}^{\infty} g_n^*$) but it is also absolutely convergent, i. e.

$$\sum_{n=1}^{\infty} \|g_n^*\|_S < \infty. \tag{7}$$

By Lemma 8 there exists a sequence of a. p. functions h_n such that

$$h_{n|A} = g_n \quad \text{and} \quad \|h_n\|_S < 3\,\|g_n^*\|_S. \tag{8}$$

Let us put $h = \sum_{n=1}^{\infty} h_n$. Obviously, in view of (7) and (8) this series is absolutely S-convergent, hence its sum h is a required extension of the given function f.

3. The notion of an almost periodic function in the sense of Riemann-Stepanoff (RS-a. p. function) was introduced by Doss [3]. An equivalent definition (see Theorem 2 in [3]) is as follows:

A real function $f(t)$ is RS-a. p. if for every $\varepsilon > 0$ there exist two Bohr a. p. functions φ and ψ such that $\varphi(t) \leqslant f(t) \leqslant \psi(t)$ for all t's and $\|\psi - \varphi\|_S < \varepsilon$.

Functions f_ξ defined in unit intervals I_ξ respectively, will be called *uniformly R-integrable*, if for every $\varepsilon > 0$ there are two families of equicontinuous functions φ_ξ and ψ_ξ such that for every ξ one has $\varphi_\xi \leqslant f_\xi \leqslant \psi_\xi$ and $\int_{I_\xi} (\psi_\xi - \varphi_\xi)\,dt < \varepsilon$.

THEOREM 4. *If $A \subset L$, $A \in \boldsymbol{I}$ and A is the union of disjoint unit intervals I_n, then in order that a measurable function f, defined on A, can be extended to an RS-a. p. function it is necessary and sufficient that the restricted functions $f_n = f|_{I_n}$ be uniformly R-integrable.*

In order to prove the necessity let us remark that if g is an RS-a. p. function, then the restricted functions $g|_I$, where I runs over all unit intervals, are uniformly R-integrable. This follows from the uniform continuity of Bohr a. p. functions φ and ψ, which is equivalent to the equicontinuity of the restricted functions $\varphi|_I$ and $\psi|_I$.

To prove the sufficiency we take two double sequences of functions $\{\varphi_n^{(m)}\}$ and $\{\psi_n^{(m)}\}$, defined on A and such that 1° for every fixed m they are equicontinuous, 2° for every m we have $\varphi_n^{(m)} \leqslant f_n \leqslant \psi_n^{(m)}$ in each I_n, 3° $\varphi_n^{(m+1)} \geqslant \varphi_n^{(m)}$ and $\psi_n^{(m+1)} \leqslant \psi_n^{(m)}$ for any m and n, 4° $\int_{I_n} (\psi_n^{(m)} - \varphi_n^{(m)})\,dt < 1/m$.

The existence of such sequences easily follows from the definition of uniform R-integrability.

Let us fix an index m. Since A is an I-set, we may find a Bohr a. p. function $\varphi^{(m)}$ equal to $\varphi_n^{(m)}$ in I_n $(n = 1, 2, \ldots)$. We may also find a Bohr a. p. function $\Delta^{(m)}$ which is equal to $\psi_n^{(m)} - \varphi_n^{(m)}$ in each I_n and non-negative. Moreover, making use of Lemma 8 we may assume that $\|\Delta^{(m)}\|_S \leqslant \leqslant 3/m$. Let $\psi^{(m)} = \varphi^{(m)} + \Delta^{(m)}$. Analogously we define a. p. functions $\varphi^{(m+1)}$ and $\psi^{(m+1)} \geqslant \varphi^{(m+1)}$ which are equal to $\varphi_n^{(m+1)}$ or $\psi_n^{(m+1)}$ respectively in each I_n and such that $\|\psi^{(m+1)} - \varphi^{(m+1)}\|_S \leqslant 3/(m+1)$. Without destroying these conditions we modify $\varphi^{(m+1)}$ and $\psi^{(m+1)}$ so as to obtain functions $\hat{\varphi}^{(m+1)}$ and $\hat{\varphi}^{(m+1)}$ fulfilling the inequalities $\varphi^{(m)} \leqslant \hat{\varphi}^{(m+1)} \leqslant \hat{\psi}^{(m+1)} \leqslant \psi^{(m)}$. This can be done in the following manner: first we put

$$\hat{\varphi}^{(m+1)}(t) = \begin{cases} \varphi^{(m)}(t) & \text{if} \quad \varphi^{(m+1)}(t) < \varphi^{(m)}(t), \\ \psi^{(m)}(t) & \text{if} \quad \varphi^{(m+1)}(t) > \psi^{(m)}(t), \\ \varphi^{(m+1)}(t) & \text{elsewhere;} \end{cases}$$

then we put

$$\hat{\psi}^{(m+1)}(t) = \begin{cases} \psi^{(m)}(t) & \text{if} \quad \psi^{(m+1)}(t) > \psi^{(m)}(t), \\ \hat{\varphi}^{(m+1)}(t) & \text{if} \quad \psi^{(m+1)}(t) < \hat{\varphi}^{(m+1)}(t), \\ \psi^{(m+1)}(t) & \text{elsewhere.} \end{cases}$$

The functions $\hat{\varphi}^{(m+1)}$ and $\hat{\psi}^{(m+1)}$ are a. p. which is easily seen by considering them, as well as the functions involved in their construction, as continuous functions on K. The inequality $\|\hat{\psi}^{(m+1)} - \hat{\varphi}^{(m+1)}\|_S \leqslant 3/(m+1)$ holds, because we have $\hat{\psi}^{(m+1)} - \hat{\varphi}^{(m+1)} \leqslant \psi^{(m+1)} - \varphi^{(m+1)}$.

Afterwards we construct $\varphi^{(m+2)}$ and $\psi^{(m+2)}$ and modify them by means of $\hat{\varphi}^{(m+1)}$ and $\hat{\psi}^{(m+1)}$ so as to get $\hat{\varphi}^{(m+2)}$ and $\hat{\psi}^{(m+2)}$, analogously as $\hat{\varphi}^{(m+1)}$ and $\hat{\psi}^{(m+1)}$ were obtained from $\varphi^{(m+1)}$ and $\psi^{(m+1)}$ by means of $\varphi^{(m)}$ and $\psi^{(m)}$. If we proceed in this manner starting from $m = 1$ we get a non-decreasing sequence of a. p. functions $\{\hat{\varphi}^{(m)}\}$ and a non-increasing sequence of a. p. functions $\{\hat{\psi}^{(m)}\}$ such that $\hat{\psi}^{(m)} \geqslant \hat{\varphi}^{(m)}$ everywhere, $\hat{\psi}^{(m)} \geqslant f_n \geqslant \hat{\psi}^{(m)}$ in all I_n's and $\|\hat{\psi}^{(m)} - \hat{\varphi}^{(m)}\|_S \leqslant 3/m$. These sequences give raise to an RS-a. p. function equal $\lim\limits_{m\to\infty} \hat{\psi}^{(m)}(t) = g(t)$, for example. An equivalent RS-a. p. function is obtained, if we modify g on L so that it remains always between $\hat{\varphi}^{(m)}$ and $\hat{\psi}^{(m)}$. So we may replace g by f_n in every I_n getting thereby the required RS-a. p. extension of the given function f on A.

4. In this section we will investigate $\boldsymbol{I}_0$-sets in locally compact Abelian groups in general. Property $\boldsymbol{I}$ could be defined in them as well as for the group L, but it does not seem to be of equal importance, e. g. in a compact group every set would be an $\boldsymbol{I}$-set, because every uniformly continuous function on a subset can be extended to a continuous and, consequently, *almost periodic* function on the whole group.

Let G be locally compact and J be a closed subgroup of G.

LEMMA 9. *If $H = G/J$ and there is an $\boldsymbol{I}_0$-set Z_1 in H, then there is also an $\boldsymbol{I}_0$-set in G, of the same cardinality as Z_1.*

Proof. If φ is the natural homomorphism $G \to H$, then, clearly, $f\varphi$ is a. p. on G provided f is a. p. on H. Therefore we obtain the required set in G by selecting one element from each coset $\varphi^{-1}(a)$ $(a \epsilon Z_1)$. (The local compactness of G was not used here).

LEMMA 10. *If Z is an $\boldsymbol{I}_0$-set in J, then it is also an $\boldsymbol{I}_0$-set in G.*

The Lemma follows at once from the fact that an a. p. function f on J can be extended to an a. p. function over G. To see this let us consider f as a continuous function on the Bohr compactification $\tilde{J}$ of J. Since $\tilde{J}$ is a closed subgroup of $\tilde{G}$, f can be continuously extended over the latter. (Obviously the Lemma would also hold with $\boldsymbol{I}$ instead of $\boldsymbol{I}_0$).

It is obvious that there is no infinite $\boldsymbol{I}_0$-set in a compact separable group G, since in this case every infinite subset of G contains a convergent sequence with distinct terms. On the other hand, there exist compact groups containing infinite $\boldsymbol{I}_0$-sets as is e. g. the group $K \supset L$ (remember that $\{2^n\}$ is an $\boldsymbol{I}_0$-set). The latter example is only a special case of much more general situation, but it is advisable to deal first with $\boldsymbol{I}_0$-sets in *discrete* groups and to prove the following

THEOREM 5. *A discrete Abelian group of cardinality $\mathfrak{m}$ contains an $\boldsymbol{I}_0$-set of the same cardinality.*

The proof will be preceded by the following

LEMMA 11. *If G is a discrete Abelian group and $Z = \{a_\iota\}$ is a set of independent elements of G (i. e. from $n_1 a_{\iota_1} + \ldots + n_k a_{\iota_k} = 0$ follows $n_j a_{\iota_j} = 0$ $(j = 1, \ldots, k)$ for any integers n_j and $k > 0$), then Z is an $\boldsymbol{I}_0$-set.*

Proof. Let us split Z into two parts: $Z = P \cup Q$. We have to prove that the (weak) closures $\tilde{P}$ and $\tilde{Q}$ in G are disjoint. To this purpose it is sufficient to point out a character χ of G such that

$$(1) \qquad |\chi(t) - \chi(s)| \geqslant \delta \quad \text{for any } t \epsilon P, \text{ any } s \epsilon Q \text{ and a fixed } \delta > 0.$$

We define a function χ_0 on the set Z as follows: if $a_\iota \epsilon P$ is of order $n < \infty$, then we put $\chi_0(a_\iota)$ equal to an arbitrary value of the n-th root of unity with negative real part; if $a_\iota \epsilon P$ is of infinite order, we put $\chi_0(a_\iota) = -1$; if $a_\iota \epsilon Q$, then $\chi_0(a_\iota) = 1$. The function χ_0 obviously can be extended to a character χ of G fulfilling (1) with $\delta = \sqrt{2}$.

Now we can immediately prove Theorem 5 for $\mathfrak{m} > \aleph_0$. In fact, it is easy to show by standard ineffective methods that G must then contain a set of $\mathfrak{m}$ independent elements. It remains to apply Lemma 11. We pass to the case $\mathfrak{m} = \aleph_0$. If G contains an element of infinite order, then it contains isomorphically the group I of integers. This group contains an infinite $\boldsymbol{I}_0$-set, e. g. the set $\{2^n\}$; in fact: every $\boldsymbol{I}_0$-set in L is $\boldsymbol{I}_0$ in I, because an a. p. function on L is also a. p. if restricted to I (a so-called *almost periodic sequence*). It remains to apply Lemma 10.

If G is a torsion group, then $G = \sum_i G_i$, where G_i are p-groups and $\sum$ denotes a direct sum. If this sum is infinite, then choose an element $a_i \neq 0$ from each G_i. The set $\{a_i\}$ is $\boldsymbol{I}_0$ by Lemma 11. If $\sum$ is finite, then one of the G_i's is infinite, and so, on account of Lemma 10, the proof has been reduced to the case of a denumerable p-group G. Such a group contains a basic subgroup B, i. e. a (pure) subgroup which is the direct sum of (finite) cyclic groups and such that G/B is divisible ([7], p. 115). If B is infinite, then Lemmas 11 and 10 imply immediately the assertion of Theorem 5. If B is finite, then G/B is infinite and so equal to the direct sum of groups C_{p^∞} ([7], p. 149). On account of Lemmas 10 and 9 the proof will be achieved if it is shown that the group C_{p^∞} contains an infinite $\boldsymbol{I}_0$-set.

Let Z be a sequence of elements $c_n \in C_{p^\infty}$ such that $c_{n+1}^{p^2} = c_n$, $c_1 = 1$. Analogously as in the proof of Lemma 11, let us split C_{p^∞} into two parts P and Q. We try to construct a character χ fulfilling (1). Since $p^2 \geqslant 4$, we can define a function χ_0 on Z step by step so as to have

$$[\chi_0(c_{n+1})]^{p^2} = \chi_0(c_n), \qquad \chi_0(c_1) = 1$$

and

$$|\chi_0(c_n) - 1| \leqslant 2\sin\frac{\pi}{8} \text{ for } c_n \in P, \qquad |\chi_0(c_n)+1)| \leqslant 2\sin\frac{\pi}{8} \text{ for } c_n \in Q.$$

The first condition is necessary and sufficient for χ_0 to be extendable from Z to a character χ of C_{p^∞}, the second ensures that χ satisfies (1) with $\delta = \sqrt{2}$.

THEOREM 6. *A compact group G whose topological weight is $\geqslant 2^{\aleph_0}$ contains an infinite $\boldsymbol{I}_0$-set.*

Proof. By Lemma 9 it is sufficient to prove that a homomorphic image of G contains an $\boldsymbol{I}_0$-set and this happens if and only if it contains homeomorphically the space $\beta(N)$ (see section 1). To contain $\beta(N)$ is a property of any Tichonoff product of at least $2^{\aleph_0}$ non trivial T_1-spaces. So it is enough to prove that there is a homomorphic image of G being such a product. Homomorphic images of G are dual groups to subgroups of the discrete group $\hat{G}$, dual to G (see [11], p. 258). As the character group of a direct sum $\sum_\iota G_\iota$ of discrete groups is the complete (Tichonoff) product of the duals $\hat{G}_\iota$, we have but to show that $\hat{G}$ contains a direct sum of $2^{\aleph_0}$ (non-trivial) groups. We have $\operatorname{card}(\hat{G}) = \mathfrak{m} \geqslant 2^{\aleph_0}$ and so $\hat{G}$ contains $\mathfrak{m}$ independent elements. Hence the direct sum generated by them satisfies our requirements. The proof is thus achieved.

We omit reasonings pertinent to the existence of infinite $\boldsymbol{I}_0$-sets in locally compact Abelian groups other than L, which are neither discrete nor compact. Corresponding theorems can be easily stated owing to the well-known structural properties of such groups.

We do not know anything about analogical questions concerning non-Abelian groups, even if they are maximally almost periodic (see [9], p. 226-227).

5. Now we give a review of some properties of sequences of reals with respect to the Bohr compactification K. The properties listed below are directly or indirectly related to the main object of our interest, i. e. to the property $\boldsymbol{I}$. An increasing sequence $Z = \{a_n\}$ is said to belong to the class

$\boldsymbol{L}$ if $a_{n+1}/a_n > q > 1$ $(n = 1, 2, \ldots)$,

$\boldsymbol{O}$ if $\mu(\tilde{Z}) = 0$, μ being the Haar measure in K,

$\boldsymbol{B}$ if $a_{n+1} - a_n < C$,

$\boldsymbol{Is}$ if Z is isolated in K,

$\boldsymbol{N}$ if Z is non-dense in K,

$\boldsymbol{D}$ if Z is dense in K,

$\boldsymbol{E}$ if Z is equidistributed in L,

$\boldsymbol{P}$ if there is a real λ such that $\|a_n\lambda\| \to 0$, $\|\cdot\|$ denoting the distance to the nearest integer.

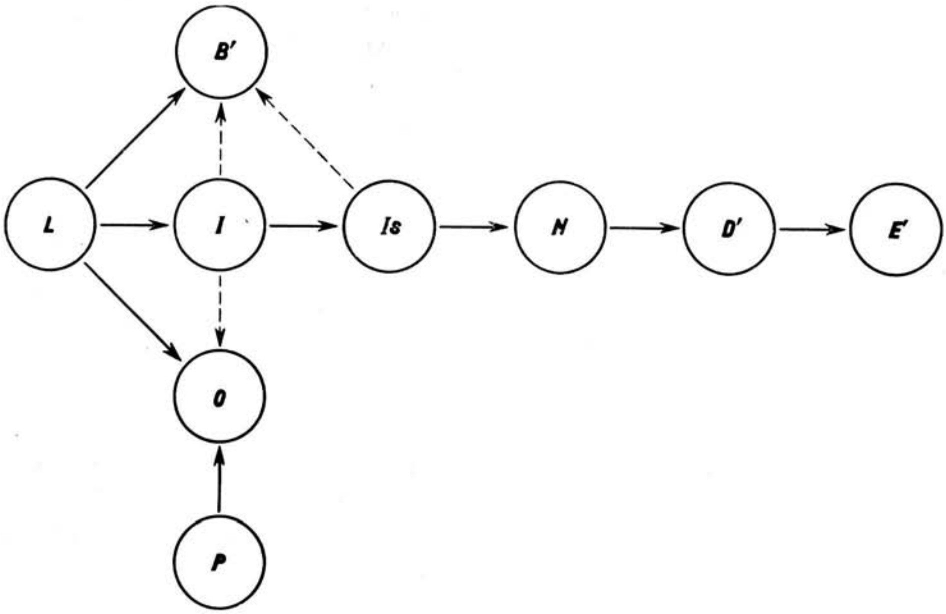
In the diagram on the next page, which involves also the class $\boldsymbol{I}$, the continuous arrows denote implication and the broken arrows indicate conjectured implications. $\boldsymbol{A}'$ means negation of the property denoted by $\boldsymbol{A}$.

THEOREM 7. *The diagram is all right. The implications converse to those indicated by continuous or by broken arrows are false, except possibly* $\boldsymbol{Is} \to \boldsymbol{I}$.

Proof. $\boldsymbol{L} \to \boldsymbol{B}'$ is trivial. $\boldsymbol{L} \to \boldsymbol{I}$ is a result of Strzelecki [13]. $\boldsymbol{L} \to \boldsymbol{O}$ is a consequence of Theorem 8 given below. $\boldsymbol{B}' \to \boldsymbol{L}$ is obviously false. $\boldsymbol{I} \to \boldsymbol{L}$ and even $\boldsymbol{I}_0 \to \boldsymbol{L}$ is false, because the sequence consisting of numbers 2^n and 2^n+1 is an $\boldsymbol{I}$-set in view of Theorem 2. The implication $\boldsymbol{O} \to \boldsymbol{L}$ is false, since the weak closure of the set I of all integers is of measure 0; in fact, otherwise, I being a group it would be the whole of the *connected* group K. But this is impossible, because there exists a periodic continuous non-zero function f with $f(n) = 0$ $(n \in I)$ and so the set N of positive integers is not dense in K. Now observe that $N \notin \boldsymbol{Is}$. This follows from the fact that there is no a. p. function with $f(1) = 0$ and

$f(n) = 1$ for $n > 1$. Since $\boldsymbol{I} \to \boldsymbol{Is}$ and $\boldsymbol{O} \to \boldsymbol{N}$ trivially hold, so $\boldsymbol{O} \to \boldsymbol{I}$ and $\boldsymbol{N} \to \boldsymbol{Is}$ turn out to be false. The implications $\boldsymbol{Is} \to \boldsymbol{N}$ and $\boldsymbol{N} \to \boldsymbol{D'}$ are trivial.

$\boldsymbol{N} \to \boldsymbol{O}$ can be disproved as follows: take a closed non-dense set E of positive Lebesgue measure on the unit circle C; the character $\chi(t) = e^{it}$, extended over K, maps K continuously and homomorphically onto this



circle. Since this homomorphism is open, the set $\chi^{-1}(E)$ is non-dense too. Clearly, χ maps the Haar measure on K onto an invariant Borel measure on C which must be then identic with the Lebesgue measure. Hence one has $\mu(\chi^{-1}(E)) > 0$. Let $M = \{z_n\}$ be a sequence dense in E. Then $\chi^{-1}(M)$ is dense in $\chi^{-1}(E)$. Obviously, each set $\chi^{-1}(z)$ $(z \in C)$ contains a positive real. So we choose a positive real number a_n $(n = 1, 2, \ldots)$ from each set $\chi^{-1}(z_n)$. Then $\chi^{-1}(M) = \bigcup_n (a_n + Q)$, where $Q = \{a \in K \colon \chi(a) = 1\}$. We claim that the set $A = \{a_n + 2\pi m\}$ $(n, m = 1, 2, \ldots)$ is dense in $\chi^{-1}(M)$. All one has to prove is that the sequence $Q_0 = \{2\pi m\}$ $(m = 1, 2, \ldots)$ is dense in Q. First of all let us observe that $\tilde{Q}_0$ is equal to the weak closure of $\{2\pi m\}$ $(m = 0, \pm 1, \pm 2, \ldots)$, because every compact subsemigroup of a group is a group. If $\tilde{Q}_0$ were a *proper* subgroup of Q, then there would exist a character equal to 1 on $\tilde{Q}_0$ but not on the whole of Q. This is, however, impossible, since a character of $\tilde{Q}_0$ which is equal to 1 on Q_0 is of the form χ^k (k integer).

So we have $\tilde{A} = \chi^{-1}(E)$, hence $A \subset L$ is non-dense in K, but $\mu(\tilde{A}) > 0$. In order to replace A by an increasing sequence we choose one point of A in the interval $(0, 1]$, if there is any, then we take one point of A in each of the intervals $(1, \frac{3}{2}]$ and $(\frac{3}{2}, 2]$, if there are any, then we choose from four intervals of length $\frac{1}{4}$ between 2 and 3 etc. We obtain an increasing sequence $\{b_n\}$ and by its construction we have $\tilde{A} \setminus \{\tilde{b}_n\} \subset L$. Hence $\mu(\{\tilde{b}_n\}) > 0$ and $\{b_n\}$ is $\boldsymbol{N}$ without being $\boldsymbol{O}$.

Property $\boldsymbol{E}$ is equivalent to the equidistribution of the sequence Z on K (see section 1). We remind that $\boldsymbol{E}$-sequences do exist, e.g. $\{n^a\}$ for any non-integer $a > 0$. Obviously $\boldsymbol{E} \to \boldsymbol{D}$. Taking $a > 1$ in $\{n^a\}$ we get a $\boldsymbol{B}'$-sequence which is not $\boldsymbol{Is}$ and so not $\boldsymbol{I}$. Therefore $\boldsymbol{B}' \to \boldsymbol{Is}$ and $\boldsymbol{B}' \to \boldsymbol{I}$ turn out to be false. $\boldsymbol{D} \to \boldsymbol{E}$ is also not true. To see this take an open set M in K whose measure is $< \frac{1}{2}$ and whose boundary is of measure 0. It follows from Weyl's equidistribution theorem that in sufficiently long intervals on the line there are points from M. So we can choose for $n > n_0$ a number from M between $n^{3/2}$ and $(n+1)^{3/2}$. We enlarge the set thus obtained by adding to it all numbers $n^{3/2}$ and so we get a sequence which is dense in K without being equidistributed. All numbers $n^{3/2}$ belonging to M form a sequence which is neither dense nor non-dense in K; so $\boldsymbol{D}' \to \boldsymbol{N}'$ is false.

Let us notice that $\boldsymbol{D}$ or $\boldsymbol{E}$ are in some sense opposite properties to $\boldsymbol{N}$, $\boldsymbol{Is}$ or $\boldsymbol{O}$. Property $\boldsymbol{P}$ has a different character. It gives no information as to the "thickness" of Z, but it may be interesting by itself, since a $\boldsymbol{P}$-sequence $Z = \{a_n\}$ differs from a subsequence of an arithmetical progression $\{m_n/\lambda\}$ $(m_n \in N)$ by a sequence tending to 0. Since $\{m_n/\lambda\}$ and Z have the same cluster points, the weak closure $\tilde{Z}$ of Z has a measure not exceeding that of an arithmetical progression. The latter is 0, which can be proved in the same way as it was done for N. So $\boldsymbol{P} \to \boldsymbol{O}$ is proved. To see that the converse does not hold we make use of the theorem of Pisot and Vijayaraghavan ([1], p. 134) stating that if for an algebraic number $\vartheta > 1$ the sequence $\{\vartheta^n\}$ has property $\boldsymbol{P}$, then ϑ is a Pisot number, i.e. an algebraic integer all conjugates of which are less than 1 in absolute value. So if we choose an algebraic number $\vartheta > 1$ not being a Pisot number, then the sequence $\{\vartheta^n\}$ does not have property $\boldsymbol{P}$, nevertheless it is an $\boldsymbol{L}$-sequence and therefore $\boldsymbol{I}$ and $\boldsymbol{O}$.

From the discussion of the diagram there arise the following questions, which the authors are not able to answer:

P 452. Is $\boldsymbol{I}_0 \to \boldsymbol{O}$ or even $\boldsymbol{I} \to \boldsymbol{O}$ true?

P 453. Is $\boldsymbol{I}_0 \to \boldsymbol{B}'$ or even $\boldsymbol{I} \to \boldsymbol{B}'$ true?

P 454. Is $\boldsymbol{Is} \to \boldsymbol{I}$ true?

P 455. Does there exist a set S with $\sigma(S) > 0$ which belongs to $\boldsymbol{N}$ and not to $\boldsymbol{O}$?

The first and second implication (but not the third one!) may be considered as the authors' conjecture. One can even guess that the weak closure not only of an increasing $\boldsymbol{I}$-sequence but of any $\boldsymbol{I}$-set in L has measure 0. This conjecture implies a partial answer on P 453: no union E of disjoint intervals of length $> l > 0$ is an $\boldsymbol{I}$-set if the distance between consecutive intervals is bounded. In fact, L can then be covered by a finite number of translations of E and so E must have positive measure, against the conjecture.

If we had used the group I of integers instead of L and the compactification $\tilde{I}$ instead of K we could build an analogous diagram and ask analogous questions. They are related to P 452-P 455; e.g. if the solution of P 452 for $\boldsymbol{I}$-sets in I is positive, then so is the solution of P 453 (by translation argument), but $\boldsymbol{I}$-sets in L consisting of integers are at the same time $\boldsymbol{I}$-sets in I and vice-versa; hence no "relatively dense" set of integers could be an $\boldsymbol{I}$-set in L.

THEOREM 8. *If E consists of disjoint, possibly degenerated intervals I_n of bounded length and $t_{n+1}/t_n > 1+a$ $(a > 0)$ for a sequence of elements $t_n \in I_n$, then E is of measure* 0.

Proof. We may assume that $|I_n| \leqslant \frac{1}{2}$. We infer from Strzelecki's theorem and from Theorem 2 that E is an $\boldsymbol{I}$-set. There is a number n_1 so that $\bigcup_{n>n_1} (I_n+1)$ is disjoint with E, and in general, for every integer $k > 0$ there is a number n_k so that $n_{k+1} \geqslant n_k$ and the sets $E_k = \bigcup_{n>n_k} (I_n+k)$ are at positive distance from one another. Obviously, for any k and $l > k$ the set $E_k \cup E_l$ consisting of I_n+k $(n = n_k+1, \ldots, n_l)$ and $(I_n+k) \bigcup (I_n+l)$ $(n > n_l)$ is contained in a union of disjoint segments J_n $(n > n_k)$ of bounded length, each containing a point t_n+k. Since $t_{n+1}/t_n > 1+a$, $E_k \cup E_l$ is an $\boldsymbol{I}$-set. Since dist $E_k, E_l) > 0$, Lemma 2 implies $\tilde{E}_k \cap \tilde{E}_l = \emptyset$. So we have proved that the closures $\tilde{E}_k$ are pairwise disjoint. Each E_k differs from $E+k$ only by a bounded set in L; hence $\mu(\tilde{E}_k) = \mu\big((E+k)^{\sim}\big) = \mu(\tilde{E}+k) = \mu(\tilde{E})$ (since the weak closure of a bounded set in L is of μ-measure 0). Thus there is a sequence of disjoint sets in K whose measures are all equal to $\mu(E)$. This proves the Theorem.

We now intend to discuss the existence of distinct subsets of K homeomorphic to the Čech space $\beta(N)$. We know that such subsets exist and it is from double source we do: the first is the existence of infinite $\boldsymbol{I}_0$-sets in L, the other is Theorem 6, since the weight of K is $2^{\aleph_0}$. Actually, we can easily prove that there are $2^{2^{\aleph_0}}$ disjoint $\beta(N)$ spaces in K. Here is the reason for it: K is the dual of the discrete group of real numbers; the latter is the direct sum of $2^{\aleph_0}$ copies of R^+ (additive group of rationals); therefore K is the Tichonoff product of $2^{\aleph_0}$ solenoidal groups $\hat{R}^+$. The assertion follows by selecting any two point set on each "axis"

R^+ and forming their Tichonoff product. In fact, every such product contains $\beta(N)$.

But these algebro-topological reasonings do not supply any knowledge about the existence of a set in K, homeomorphic to $\beta(N)$ and such that its isolated points are all in L. Only such knowledge would be sufficient to deduce the existence of $\boldsymbol{I}_0$-sets in L without special theorems of arithmetical kind, like the quoted theorem of Strzelecki [13]. On the other hand, Strzelecki's theorem enables us to show that in K there are $2^{\aleph_0}$ disjoint copies of $\beta(N)$ such that their isolated points lie all in L. To see this observe that the union of any two of the sequences $\{C\cdot 2^n\}$ $(1\leqslant C<2)$ has property $\boldsymbol{L}$ and so it is an $\boldsymbol{I}_0$-set. By property $\boldsymbol{S}_0$ the weak closures of any two such sequences are disjoint. Since each of these closures is $\beta(N)$, our assertion is proved. We notice still more: the copies of $\beta(N)$ we have just obtained are transformable one onto the other by a group operation in K (since multiplying by a real is such operation). Finally we put the following question:

P 456. If the sequences $\{\alpha^n\}$ and $\{\beta^n\}$ $(\alpha>1,\ \beta>1)$ are "far one from another" [6], i.e. if for any M we have $|\alpha^m-\beta^n|>M$ for $m,n>n_0(M)$, must then the union of these sequences be an $\boldsymbol{I}$-set?

Let us remark that this union is never an L-sequence and that $\{\alpha^n\}$ and $\{\beta^n\}$ are far one from another e.g. at the case α and β are distinct primes, which is a consequence of a theorem of Gelfond ([4], p. 40).

REFERENCES

[1] J. W. S. Cassels, *An introduction to diophantine approximation*, Cambridge 1957.

[2] J. Cigler, *Folgen normierter Maße auf kompakten Gruppen*, Zeitschrift für Wahrscheinlichkeitstheorie 1 (1962), p. 3-13.

[3] R. Doss, *On Riemann integrability and almost periodic functions*, Compositio Mathematica 12 (1954-1956), p. 271-283.

[4] А. О. Гельфонд, *Приближение алгебраических чисел алгебраическими же числами и теория трансцендентных чисел*, Успехи Математических Наук IV, 4 (1949), p. 19-49.

[5] S. Hartman, *On interpolation by almost periodic functions*, Colloquium Mathematicum 8 (1961), p. 99-101.

[6] — *Sur une relation entre les suites*, ibidem 2 (1951), p. 325.

[7] А. Г. Курош, *Теория групп*, Москва 1953.

[8] J. S. Lipiński, *Sur un problème de E. Marczewski concernant les fonctions périodiques*, Bulletin de l'Académie Polonaise des Sciences, Série math., astr. et phys., 8 (1960), p. 695-697.

[9] W. Maak, *Fastperiodische Functionen*, Berlin-Göttingen-Heidelberg 1950.

[10] J. Mycielski, *On a problem of interpolation by periodic functions*, Colloquium Mathematicum 8 (1961), p. 95-97.

[11] А. С. Понтрягин, *Непрерывные группы*, Москва 1954.

[12] C. Ryll-Nardzewski, *Remark on interpolation by periodic functions*, Bulletin de l'Académie Polonaise des Sciences, Série Math., Astr. et Phys., 11 (1963), p. 363-366.

[13] E. Strzelecki, *On a problem of interpolation by periodic and almost periodic functions*, Colloquium Mathematicum 11 (1963), p. 91-99.

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