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DIRECT LIMITS OF TOPOLOGICAL SPACES AND GROUPS

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The purpose of this paper is to define the direct limit of completely regular topological spaces and the direct limit of topological groups. The first problem has a solution in a paper of Vegrin [6]. The problem of introducing at once topology and group operation into direct limit set was considered by Chogoshvili [2] and Berikashvili [1] and was solved for direct systems consisting only of compact or only of discrete groups, by using the duality theory of topological groups.

At the beginning, we recall the notion of direct system and direct limit in general categories. In section 2 we introduce a notion of the closure of an equivalence relation. In the case of topological compact spaces this notion, which is in fact dual to the notion of the closure of subsets, is connected in a natural manner with topology. We define the drect limit of completely regular topological spaces as the quotient space of Čech compactification of the disjoint sum of spaces of direct system by an equivalence relation which is the closure of the equivalence relation induced by mappings of the direct system. We show in section 5 that in the case when the objects of the system are topological groups, the limit defined in this way admits a continuous group operation.

1. Direct systems and direct limits in general categories. Let $\mathscr{X}$ be a category (1). Let M be a directed set considered as a category: objects of M are elements of (set) M, mappings are ordered pairs $\alpha < \beta$, $\alpha, \beta \in M$. An arbitrary covariant functor $\pi\colon M \to \mathscr{X}$ is said to be *direct system* in category $\mathscr{X}$.

Direct systems form a category if $\mathrm{Map}(\pi, \pi')$, $\pi\colon M \to \mathscr{X}$, $\pi'\colon M' \to \mathscr{X}$, consists of mappings defined as follows. Take a mapping (functor) p: $M \to M'$ such that $\alpha < \beta \Rightarrow p\alpha < p\beta$ and define $f\colon \pi \to \pi'$ as a mapping

(1) For the notions of category, functor etc. see Godement [4].

of functors $\pi \to \pi' p$, i.e. define f as a system of mappings $f_a\colon X_a \to X'_{pa}$, $a \in M$, with commutativity property

$$\begin{array}{ccc} X_a & \longrightarrow & X'_{pa} \\ \downarrow & & \downarrow \\ X_\beta & \longrightarrow & X'_{p\beta} \end{array}$$

for $a, \beta \in M$ and $a < \beta$. Here X_a and X_β denote values of π at a and β respectively (they should be denoted, in fact, by $\pi(a)$ and $\pi(\beta)$), and this notation will be used in the sequel. Mappings $\pi(a < \beta)$ will be denoted by π^a_β.

Consider a covariant functor from category $\Pi(\mathscr{X})$ of direct system in $\mathscr{X}$ into category $\mathscr{X}$. Such a functor is said to be *direct limit* in $\mathscr{X}$, $\varinjlim\colon \Pi(\mathscr{X}) \to \mathscr{X}$, if the following properties are satisfied (see Grothendieck [5]):

(1) there exist mappings $\pi_a\colon X_a \to \varinjlim \pi$, $a \in M$, such that $a < \beta \Rightarrow$ $\Rightarrow \pi_a = \pi_\beta \pi^a_\beta$;

(2) if Y is an object of $\mathscr{X}$ and $u_a\colon X_a \to Y$, $a \in M$, are mappings for which

(i) $$u_\beta \pi^a_\beta = u_a$$

holds for every $a, \beta \in M$, $a < \beta$, then there exists a mapping $u\colon \varinjlim \pi \to Y$ uniquely determined by u_a, $a \in M$, such that

(ii) $$u\pi_a = u_a.$$

It is well known that functor $\varinjlim$ is determined uniquely, up to an isomorphism, by conditions (1) and (2) (see Grothendieck [5] and compare with Berikashvili [1]).

2. Semicontinuity. Let X be a set and $\mathscr{R}$ an equivalence relation on it. By the same letter $\mathscr{R}$ we denote the decomposition induced by $\mathscr{R}$ on X. Consider partial ordering of equivalence relations:

$$\mathscr{R}' < \mathscr{R}'' \Leftrightarrow (x\mathscr{R}'y \Rightarrow x\mathscr{R}''y).$$

Denote by $\mathscr{R}^0$ the identity and by $\mathscr{R}^1$ the relation which holds for every pair of elements of X. We have $\mathscr{R}^0 < \mathscr{R} < \mathscr{R}^1$ for every $\mathscr{R}$.

1. *Intersection* of $\mathscr{R}^a$, $a \in A$, is a relation $\mathscr{R} = \bigcap_{a \in A} \mathscr{R}^a$ defined by

$$x\mathscr{R}y \Leftrightarrow x\mathscr{R}^a y \text{ for every } a \in A.$$

2. *Sum* of $\mathscr{R}^a$, $a \in A$, is a relation $\mathscr{R} = \bigcup_{a \in A} \mathscr{R}^a$ defined as a minimal equivalence relation such that

$$x\mathscr{R}^a y \Rightarrow x\mathscr{R}y \text{ for an } a \in A.$$

3. *Cartesian product.* Let X_a, $a \in A$, be sets and $\mathscr{R}^a$ equivalence relations on X_a. The *Cartesian product* is a relation $\mathscr{R} = \mathbf{P}_{a \in A} \mathscr{R}^a$ on $X = \mathbf{P}_{a \in A} X_a$ defined by

$$x\mathscr{R}y \Leftrightarrow x_a \mathscr{R}^a y_a \text{ for every } a \in A.$$

In the case of two equivalence relations $\mathscr{R}, \mathscr{T}$ we use notation $\mathscr{R} \frown \mathscr{T}$, $\mathscr{R} \smile \mathscr{T}$ and $\mathscr{R} \times \mathscr{T}$.

Remark. Subsets are special cases of decompositions, namely, subset $A \subset X$ may be considered as a decomposition $\mathscr{A}$ given by

$$x\mathscr{A}y \Leftrightarrow x, y \in A \text{ or } x = y.$$

The intersection of subsets considered as relations coincides with intersection of subsets in usual set-theoretical sense.

A family $\boldsymbol{R}$ of equivalence relations on X is said to be *semicontinuity* on X if

D1. $\mathscr{R}^1 \in \boldsymbol{R}$,

D2. $\mathscr{R}^a \in \boldsymbol{R} \Rightarrow \bigcap_{a \in A} \mathscr{R}^a \in \boldsymbol{R}$ for $a \in A$.

Elements of $\boldsymbol{R}$ we call *semicontinuous decompositions* of X.

Let $\mathscr{R}$ be a decomposition of X. Denote by $\mathscr{R}^*$ a decomposition of X which is the intersection of all semicontinuous decompositions $\mathscr{R}'$ of X such that $\mathscr{R} < \mathscr{R}'$. By D2, $\mathscr{R}^*$ is semicontinuous. We have

C1. $\mathscr{R} < \mathscr{R}^*$,

C2. $\mathscr{R}^{1*} = \mathscr{R}^1$,

C3. $\mathscr{R}^{**} = \mathscr{R}^*$.

C4. $(\mathscr{R} \smile \mathscr{T})^* > \mathscr{R}^* \smile \mathscr{T}^*$.

The inequality converse to C4 is not true in general.

Remark. Denote by $A_{\mathscr{R}}$ the sum of elements $R \in \mathscr{R}$ such that $A \frown R \neq 0$. Let X be a compact space. We take as semicontinuous decompositions of X all these decompositions $\mathscr{R}$ of X with the property: $A \subset X$ is closed $\Rightarrow A_{\mathscr{R}}$ is closed. It is easy to verify that conditions D1 and D2 are satisfied.

Therefore, the semicontinuity of a compact space X is canonically determined by the topology of X.

Let Y be a subspace of X and $\boldsymbol{R}$ a semicontinuity on X. Denote by $\boldsymbol{R}_Y$ the family of decompositions of Y induced by $\boldsymbol{R}$ as follows: if $\mathscr{R} \in \boldsymbol{R}$, then the induced decomposition $\mathscr{R}_Y \in \boldsymbol{R}_Y$ is given by

$$y' \mathscr{R}_Y y'' \Leftrightarrow i(y') \mathscr{R} i(y''),$$

where $i: Y \to X$ is the inclusion map. It is easy to verify that $\boldsymbol{R}_Y$ satisfies the conditions D1 and D2, i.e. that $\boldsymbol{R}_Y$ is a semicontinuity on Y.

We conclude that if Y is a topological space and X is a compactification of Y, then this compactification induces canonically a semicontinuity on Y.

3. Properties of mappings with respect to semicontinuity induced by compactification. Let X be compact and $f\colon X \to Y$ continuous. It is well known that the decomposition of X into $f^{-1}(y)$, $y \in Y$, is semicontinuos, as it satisfies the property of the second remark of section 2.

We say that f *annules* $\mathscr{R}$ if, for every $R \in \mathscr{R}$, $f(R)$ is a single point.

Let $Y \subset X$ and let $\mathscr{R}_Y$ be a decomposition of Y. Relation $\mathscr{R}_Y$ induces a relation $\mathscr{R}_X$ on X given by $a\mathscr{R}_X b \Leftrightarrow a, b \in Y$ and $a\mathscr{R}_Y b$. Consider the closure $\mathscr{R}_X^*$ of the relation $\mathscr{R}_X$. Relation $\mathscr{R}_X^*$ induces in a natural manner a relation in Y, which will be denoted by $\mathscr{R}_{Y\subset X}^*$ and will be called the *closure of* $\mathscr{R}_Y^*$ *with respect to the inclusion* $Y \subset X$.

We prove some properties of continuous mappings with respect to the semicontinuity induced by β-extensions.

PROPERTY 1. *If X, Y are completely regular, $f\colon X \to Y$ continuous and f annules $\mathscr{R}$, then f annules $\mathscr{R}^*$, where $\mathscr{R}^*$ is the closure of $\mathscr{R}$ with respect to the inclusion $X \to \beta X$.*

Proof. Consider the diagram

$$\begin{array}{ccc} X & \longrightarrow & \beta X \\ \downarrow f & & \downarrow f_* \\ Y & \longrightarrow & \beta Y \end{array}$$

where f_* is the Čech extension of f (see [3]). Denote by the same letter $\mathscr{R}$ the relation induced by $\mathscr{R}$ on βX. Let $\mathscr{R}^*$ denote the closure of $\mathscr{R}$ in $\beta(X)$. It is sufficiently to prove that f_* annules $\mathscr{R}^*$, i.e. $\{f_*^{-1}(y)\}_{y\in\beta(Y)} > \mathscr{R}^*$. This inequality follows from inequality $\{f_*^{-1}(y)\}_{y\in\beta Y} > \mathscr{R}$, which is true by hypothesis, and from the fact that $\{f_*^{-1}(y)\}_{y\in\beta Y}$ is semicontinuous.

PROPERTY 2. *Let $X \overset{i_X}{\to} \beta X \overset{\eta_X}{\to} \beta X/\mathscr{R}^*$ be a sequence of mappings, where $\beta X/\mathscr{R}^*$ is the quotient space and η_X is the natural mapping (the topology in $\beta X/\mathscr{R}^*$ is the finest topology by which η_X is continuous). Let $f\colon X \to Y$, where Y is completely regular, be continuous and annulating $\mathscr{R}$. Then the sequence may be extended to the diagram*

$$\begin{array}{ccccc} X & \xrightarrow{i_X} & \beta X & \xrightarrow{\eta_X} & \beta X/\mathscr{R}^* \\ \downarrow f & & \downarrow f_* & \swarrow & \\ Y & \xrightarrow{i_y} & \beta Y & f_{**} & \end{array}$$

Proof. It remains to define the mapping f_{**} and to prove the continuity of it. Let $r = \eta_X(x)$. We take $f_{**}(r) = f_*(x)$. The mapping f_{**} is well defined, as all x in question lie in the same element R of $\mathscr{R}^*$, and f_*, by Property 1, annules $\mathscr{R}^*$. The continuity of f_{**} follows from the implication ($fg = h$, g and h continuous) $\Rightarrow$ (f continuous), which is true for mappings of quotient spaces.

Let $\mathscr{R}$ and $\mathscr{T}$ be decompositions of X and Y, and let $(\mathscr{R}\times\mathscr{T})^*$ be the closure of $\mathscr{R}\times\mathscr{T}$ with respect to the inclusion map $i_{X\times Y}\colon X\times Y \to \beta(X\times Y)$. Let $\eta_{X\times Y}\colon \beta(X\times Y)\to\beta(X\times Y)/(\mathscr{R}\times\mathscr{T})^*$ be the natural mapping.

PROPERTY 3. *There exists canonical homeomorphism* $\eta_{X\times Y} i_{X\times Y}(X\times Y) \sim \eta_X i_X(X)\times \eta_Y i_Y(Y)$.

Proof. Consider commutative diagram

$$\begin{array}{ccccc} X\times Y & \longrightarrow & \beta(X\times Y) & \longrightarrow & \beta(X\times Y)/(\mathscr{R}\times\mathscr{T})^* \\ & \searrow^{f} & \downarrow f_* & & \downarrow f_{**} \\ & & \beta X\times\beta Y & \longleftrightarrow & \beta X\times\beta Y/\mathscr{R}^*\times\mathscr{T}^*, \end{array}$$

where $f(x, y) = \big(i_X(x), i_Y(y)\big)$ and f_* is the Čech extension of f.

Note that $\eta f_* i_{X\times Y}$ annules relation $(\mathscr{R}\times\mathscr{T})^*$. In fact, relation on $\beta(X\times Y)$, which is the counterimage f_*^{-1} of the relation $\mathscr{R}^*\times\mathscr{T}^*$ on $\beta(X)\times\beta Y$, is closed. It contains the relation induced by $i_{X\times Y}$ and $\mathscr{R}\times\mathscr{T}$. Therefore it contains $(\mathscr{R}\times\mathscr{T})^*$. This implies, by Property 2, the existence of f_{**}.

It is easily verified that f is a homeomorphism, relations on $X\times Y$ induced by $(\mathscr{R}\times\mathscr{T})^*$ on $\beta(X\times Y)$ and by $\mathscr{R}^*\times\mathscr{T}^*$ on $\beta X\times\beta Y$ are equal. Hence mapping $f_{**}\,|\,\eta_{X\times Y} i_{X\times Y}(X\times Y)$ is a homeomorphism. It maps canonically $\eta_{X\times Y} i_{X\times Y}(X\times Y)$ onto $\eta_X i_X(X)\times\eta_Y i_Y(Y)$.

4. Direct limit of topological spaces. Let $\mathscr{X}$ be the category of completely regular topological spaces and continuous mappings. Consider a direct system in $\mathscr{X}$, i.e. $\pi\colon M\to\mathscr{X}$.

Let S be a disjoint sum of X_α, $\alpha\in M$, with a topology in which every X_α is closed-open and which coincides on X_α with a given (completely regular) topology of X_α. Let $\sim$ be an equivalence relation given by

$$x_\alpha \sim x_\beta \Leftrightarrow \text{there exists } \gamma\colon\ \pi_\gamma^\alpha(x_\alpha) = \pi_\gamma^\beta(x_\beta).$$

Consider the sequence of mappings

$$S \xrightarrow{i} \beta S \xrightarrow{\eta} \beta S/\sim^*,$$

where $\sim^*$ is taken with respect to βS.

We define

$$\varinjlim \pi = \eta i(S).$$

We prove that $\varinjlim\pi$ satisfies axioms of section 2.

1. $\varinjlim$ is a functor. Let $f\colon \pi \to \pi'$ be a mapping of direct systems $\pi\colon M \to \mathscr{X}$ and $\pi'\colon M' \to \mathscr{X}$ given by $f_a\colon X_a \to X'_{pa}$, where $p\colon M \to M'$. In order to define the mapping $\varinjlim f\colon \varinjlim \pi \to \lim \pi'$, consider mapping $\varphi\colon S \to S'$ given by $\varphi \mid X_a = f_a$. Consider the diagram

$$\begin{array}{ccccc} S & \xrightarrow{i} & \beta S & \xrightarrow{\eta} & \beta S/\sim^* \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi_*} & & {\scriptstyle\varphi_{**}}\downarrow \\ S' & \xrightarrow{i'} & \beta S' & \xrightarrow{\eta'} & \beta S'/\sim'^* \end{array}$$

To prove the existence, note that, by commutativities $f_\beta \pi^a_\beta = \pi'^{pa}_{p\beta} f_a$, $a, \beta \in M$, mapping $\eta' i' \varphi$ annules $\sim$. Hence, by Property 1 and the uniqueness of Čech extension of mappings, mapping $\eta' \varphi_*$ annules $\sim^*$.

The existence of φ_{**} and commutativity of the second rectangle of the diagram is a consequence of Property 2.

In particular, we have

$$\varphi_{**}\eta i(S) \subset \eta' i'(S').$$

Hence, we define

$$\varinjlim f = \varphi_{**} \mid \eta i(S).$$

Now it is easy to verify that $\varinjlim$ is a functor, by showing that $\varinjlim e_\pi = e_{\varinjlim \pi}$ and $\varinjlim fg = \varinjlim f \varinjlim g$ (proofs of these formulas will be omitted).

Introduce projections $\pi_a\colon X_a \to \varinjlim \pi$ by formula

$$\pi_a = \eta i \mid X_a.$$

By definition and commutativity of the diagram, we have

$$(\varinjlim f)\pi_a = \pi'_{pa} f_a.$$

Proof of (1). Let $x_a \in X_a$. We have $x_a \sim \pi^a_\beta(x_a)$. This implies $x_a \sim^* \pi^a_\beta(x_a)$ and, in consequence, $i\eta(x_a) = i\eta\pi^a_\beta(x_a)$, i.e. $\pi_a(x_a) = \pi_\beta \pi^a_\beta(x_a)$.

Proof of (2). Let $u_a\colon X_a \to Y$, where Y is completely regular, be continuous. Consider $u'\colon S \to Y$ given by $u' \mid X_a = u_a$. By (i), u' annules $\sim$, hence, by Property 2, we have the commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{i} & \beta S & \xrightarrow{\eta} & \beta S/\sim^* \\ \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u'_*} & \swarrow{\scriptstyle u'_{**}} & \\ Y & \xrightarrow{j} & \beta Y & & \end{array}$$

Now, we define u by formula

$$u = u'_{**} \mid \eta i(S).$$

Commutativity (ii) follows from commutativity of the diagram.

5. Direct limit of topological groups. Let X_a, $a \in M$, be topological (completely regular) groups and $\pi^a_\beta\colon X_a \to X_\beta$, $a < \beta$, continuous homomorphisms. We shall introduce group operation in the space $\underrightarrow{\lim}\,\pi$ defined above and we shall prove that functor $\underrightarrow{\lim}$ considered on category of topological groups is a direct limit in this category, i.e. π_a and $\underrightarrow{\lim} f$ are homomorphisms.

Let $\varphi_a\colon X_a \times X_a \to X_a$ be group operation on X_a. In order to define group operation $\varphi\colon \underrightarrow{\lim}\,\pi \times \underrightarrow{\lim}\,\pi \to \underrightarrow{\lim}\,\pi$, consider continuous mapping $v\colon S \times S \to \underrightarrow{\lim}\,\pi$ defined as follows: let $(x_a, x_\beta) \in X_a \times X_\beta \subset S \times S$ and let $\gamma > a$ and $\gamma > \beta$; we define $v(x_a, x_\beta) = \eta i \varphi_\gamma\big(\pi^a_\gamma(x_a), \pi^\beta_\gamma(x_a)\big)$, and this is meaningful, for the right side does not depend on γ. Now, we define

$$\varphi' = v_{**} \mid \eta_{S\times S} i_{S\times S}(S \times S),$$

where v_{**} corresponds to v in the same manner as u_{**} to u in the diagram of Property 2. Property 2 implies the continuity of φ'. The composition

$$\eta i(S) \times \eta i(S) \overset{h}{\sim} \eta_{S\times S} i_{S\times S} S \times S \xrightarrow{v_{**}} \eta i(S) = \underrightarrow{\lim}\,\pi,$$

where h is the canonical homeomorphism of Property 3 for $X = Y = S$, is the required group operation on $\underrightarrow{\lim}\,\pi = \eta i(S)$. In the formulas which follow we write often $+$ instead of φ and φ_a.

Neutral element is defined as the image ηi of neutral elements of groups X_a. The inverse of $x \in \underrightarrow{\lim}\,\pi$ is defined as a mapping $\underrightarrow{\lim}\,\pi \to \underrightarrow{\lim}\,\pi$ induced by the mapping $S \to S$ which transforms $x_a \in S$ onto $-x_a$. The continuity of the inverse operation follows from Property 2.

Projections π_a are homomorphisms. In fact, let $x_a, y_a \in X_a$. We have $\pi_a(x_a + y_a) = \eta i \varphi_a(x_a, y_a) = v_{**}\eta_{S\times S} i_{S\times S}(x_a, y_a) = \varphi' \eta_{S\times S} i_{S\times S}(x_a, y_a)$ $= \varphi' h[\eta_S i_S(x_a), \eta_S i_S(y_a)] = \eta_S i_S(x_a) + \eta_S i_S(y_a) = \pi_a(x_a) + \pi_a(y_a)$.

Also $\underrightarrow{\lim} f$ is a homomorphism. In fact, let $x, y \in \lim \pi$. We have $x = \pi_a(x_a)$, $y = \pi_a(y_a)$ for an $a \in M$. The calculation which follows is obvious if we consider the following commutative diagram:

$$\begin{array}{ccccccc}
S \times S & \xrightarrow{i_{S\times S}} & \beta(S\times S) & \xrightarrow{\eta_{S\times S}} & \beta(S \times S)/(\sim\times\sim)^* & \xleftarrow{h} & \eta i(S) \times \eta i(S) \\
\downarrow & & \downarrow & & \downarrow & & \\
S & \xrightarrow{i} & \beta S & \xrightarrow{\eta} & \beta S/\sim^* & & \\
\downarrow & & \downarrow & & \downarrow & & \\
S' & \xrightarrow{i'} & \beta S' & \xrightarrow{\eta'} & \beta S'/\sim'^* & &
\end{array}$$

Hence we have $\underrightarrow{\lim} f(x+y) = \eta' i f_a(x_a+y_a) = \eta' i' [f_a(x_a)+f_a(y_a)] = \pi'_{pa}[f_a(x_a)+f_a(y_a)] = \pi'_{pa} f_a(x_a) + \pi'_{pa} f_a(y_a) = \underrightarrow{\lim} f\pi_a(x_a) + \underrightarrow{\lim} f\pi_a(y_a) = \underrightarrow{\lim} f(x) + \underrightarrow{\lim} f(y)$.

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