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SEPARABILITY AND $\aleph_1$-COMPACTNESS

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A space is said to be $\aleph_1$-*compact* provided that every uncountable subset of it has a limit point. F. B. Jones has shown that if $2^{\aleph_0} < 2^{\aleph_1}$, then every separable normal Fréchet space-L is $\aleph_1$-compact ([3], Theorem 3). Theorem 1 of this paper * establishes the converse, namely that if $2^{\aleph_0} = 2^{\aleph_1}$, then there is a separable normal Fréchet space-L which is not $\aleph_1$-compact.

A corollary to Jones's theorem is that if $2^{\aleph_0} < 2^{\aleph_1}$, then every separable normal Moore space is metrizable ([3], Theorem 5). The question of whether the hypothesis that $2^{\aleph_0} < 2^{\aleph_1}$ can be removed from that theorem is not settled in this paper; however, in the hope that they might be useful towards that end, two metrization theorems for separable spaces are established independent of any form of the continuum hypothesis.

Any terms not defined in this paper are as defined in [3], [4], or [7]. In particular a Moore space is a space satisfying Axiom 0 and the first three parts of Axiom 1 of [7].

The proof of Theorem 1 makes use of the following lemma from [8], p. 410, in which N denotes the set of all positive integers and c denotes the power of the continuum.

LEMMA. *There exists a family $\mathscr{F}$ of subsets of N which has c members, such that $F_1 \cap \ldots \cap F_n \cap (N - F_{n+1}) \cap \ldots \cap (N - F_m) \neq \emptyset$ for every finite collection of distinct sets $F_i \in \mathscr{F}$ $(i = 1, \ldots, m)$.*

THEOREM 1. *If $2^{\aleph_0} = 2^{\aleph_1}$, then there exists a separable normal T_1-space which is not $\aleph_1$-compact.*

Proof. A separable normal T_1-space S which is not $\aleph_1$-compact is constructed as follows. Let X denote the x-axis of the Cartesian plane E^2 and let M be a subset of X of cardinality $\aleph_1$. For each positive integer n, let H_n be the set of all points of E^2 with rational first coordinate and second coordinate equal to $1/n$. By the above lemma and the assumption

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that $2^{\aleph_1} = 2^{\aleph_0} = c$, there is a one-to-one function f from the set of all subsets of M into the set of all infinite unions of the H_i such that, for every $A \subset M$, $f(A) \cap f(M-A) = \emptyset$ and such that if $M_1, M_2, \ldots, M_m$ is a finite collection of distinct subsets of M, $M_i \neq M - M_j$ for $i, j = 1, 2, \ldots, m$, then

$$f(M_1) \cap \ldots \cap f(M_n) \cap f(M - M_{n+1}) \cap \ldots \cap f(M - M_m) \neq \emptyset.$$

Now let the points of S be $M \cup (\bigcup_{i=1}^{\infty} H_i)$ and let a basis G for S consist of all finite intersections of sets of the following three types: (1) For each subset M_0 of M, $f(M_0) \cup M_0$ is a member of G, (2) every set consisting of a single point of $\bigcup_{i=1}^{\infty} H_i$ is a member of G, and (3) for every point p of M and every open disc C in the upper half plane and tangent to X at p, $\{p\} \cup [C \cap S]$, is an open set.

The space S is separable, since the countable set $\bigcup_{i=1}^{\infty} H_i$ is dense in S. The space is also normal. For suppose that A and B are disjoint closed sets in S. Then, since $[A - f(A \cap M)] \cap (S-M)$ and $[B - f(M - A \cap M)] \cap (S - M)$ are open, $[A \cup f(A \cap M) \cap [S - B]$ and $[f(M - A \cap M) \cup B] \cap [S - A]$ are disjoint open sets containing A and B respectively. Finally S is not $\aleph_1$-compact, since the uncountable set M has no limit point.

The author has been unable to determine whether there exists a function f satisfying the above conditions and such that the space S would be a Moore space — or would even satisfy the first axiom of countability.

Definition 1. A T_2-space S is *semi-metric* provided that there exists a distance function d for S such that (1) if each of x and y is a point of S, then $d(x, y) = d(y, x) \geqslant 0$, (2) $d(x, y) = 0$ if and only if $x = y$, and (3) the topology of S is invariant with respect to d.

Notation. For a point p of S and $c > 0$, $U_c(p)$ will denote the circular neighborhood $\{x : d(x, p) < c\}$ of radius c about p. Condition (3) above then says that: if p is a point of the open set R, then there is a $c > 0$ such that $U_c(p) \subset R$; and for each $c > 0$ there is an open set Q such that $p \in Q \subset U_c(p)$.

Definition 2. The space S is *strongly complete* provided that there exists a distance function d for S satisfying conditions (1), (2) and (3) of Definition 1 and having the additional property that (4) if $M_1, M_2, \ldots$ is a decreasing sequence of closed sets, and, for each n, there is a point p_n such that $M_n \subset U_{1/n}(p_n)$, then $\bigcap_{n=1}^{\infty} M_n \neq \emptyset$.

Note that every Moore space is a regular semi-metric space (see [2] and [5]) but that the converse is not true. Theorem 2 is a generalization of Theorem 2.2 of [5].

THEOREM 2. *Every separable strongly complete regular semi-metric space is $\aleph_1$-compact and hence metrizable.*

Proof. Suppose that S is a separable, strongly complete, regular semi-metric space. Then S is $\aleph_1$-compact. For suppose that M is an uncountable subset of S having no limit point.

Let H be a countable dense subset of S. Define the decreasing sequence $M_1, M_2, \ldots$ of subsets of M as follows.

Since H is countable and for every point m of M, $U_1(m)$ must contain a point of H, then there must be a point h_1 of H such that $M_1 = \{x: x \in M, d(x, h_1) < 1\}$ is uncountable. Similarly, for each $n > 1$, there must be a point h_n such that $M_n = \{x: x \in M_{n-1}, d(x, h_n) < 1/n\}$ is uncountable. If $\bigcap_{i=1}^{\infty} M_i$ contains a point p, then for each n, $d(p, h_n) < 1/n$; hence, since S is a T_2-space, $\bigcap_{i=1}^{\infty} M_i$ can contain at most one point. Therefore, for each n, $M_n - \bigcap_{i=1}^{\infty} M_i \neq \emptyset$, $(M_n - \bigcap_{i=1}^{\infty} M_i) \subset U_{1/n}(h_n)$, and $\bigcap_{n=1}^{\infty} [M_n - \bigcap_{i=1}^{\infty} M_i] = \emptyset$ contrary to S being strongly complete. Thus S is $\aleph_1$-compact. Since every strongly complete regular semi-metric space is a Moore space ([2], Theorem 4.2), and an $\aleph_1$-compact Moore space is metrizable ([3], Lemma C), then S is metrizable.

Theorem 3 is a generalization of theorems due to McAuley and Bing ([6], Theorem 3, and [1], Theorem 5), since property P as defined below is clearly a generalization of pointwise paracompactness and screenability.

Definition 3. The space S has property P provided that every open covering G of S has an open refinement H such that H covers S and no point of S belongs to more than countably many members of H.

THEOREM 3. *Every separable space S having property P is $\aleph_1$-compact, and hence, if S is also a Moore space, S is metrizable.*

Proof. The space S is $\aleph_1$-compact. For suppose that M is an uncountable subset of S having no limit point. Let K be a countable dense subset of S and, for each point x of M, let $R(x)$ be an open set containing x and no point of $M - \{x\}$.

Then, if G is the open covering of S consisting of $S - M$ and $\{R(x): x \in M\}$, any refinement of G must include a subset Q such that, for each x and y in M, $x \neq y$, there are members g and h of Q such that $x \in g$, $y \in h$ and $g \neq h$.

Then since every member of Q must contain a point of the countable set K there is a point z in K such that z belongs to uncountably many members of Q contrary to the assumption that S has property P. Again, if the $\aleph_1$-compact space S is a Moore space, then it is metrizable by [3], Lemma C.

REFERENCES

[1] R. H. Bing, *Metrization of topological spaces*, Canadian Journal of Mathematics 3 (1951), p. 175-186.

[2] R. W. Heath, *Arc-wise connectedness in semi-metric spaces*, Pacific Journal of Mathematics 12 (1963), p. 1301-1319.

[3] F. B. Jones, *Concerning normal and completely normal spaces*, Bulletin of the American Mathematical Society 43 (1937), p. 671-677.

[4] J. L. Kelley, *General topology*, Princeton 1955.

[5] L. F. McAuley, *A relation between perfect separability, completeness, and normality in semi-metric spaces*, Pacific Journal of Mathematics 6 (1956), p. 315-326.

[6] — *A note on complete collectionwise normality and paracompactness*, Proceedings of the American Mathematical Society 9 (1958), p. 769-799.

[7] R. L. Moore, *Foundations of point set theory*, American Mathematical Society Colloquium Publications 13 (1932).

[8] W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Mathematical Journal 23 (1956), p. 409-419.

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