RELATIVE PROCESSES
WITH CONTINUOUS DISTRIBUTION FUNCTIONS

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1. Introduction. Let \( E \) be a Lebesgue measurable subset of the positive half-line. By \( |E| \) we shall denote the Lebesgue measure of \( E \). The limits

\[
|E|_R = \lim_{T \to \infty} \frac{1}{T} |E \cap [0, T)|, \quad |E|_R = \lim_{T \to \infty} \frac{1}{T} |E \cap [0, T)|
\]

are called the lower relative measure of \( E \) and the upper relative measure of \( E \) respectively. If \( |E|_R = |E|_R \), the set \( E \) is said to be relatively measurable; its lower and upper relative measures are then called simply relative measures and denoted by \( |E|_R \). Obviously, the complement \( E' \) of a relatively measurable set \( E \) is also relatively measurable and \( |E'|_R = 1 - |E|_R \). Moreover, if \( E_1 \subset E_2 \) and both \( E_1 \) and \( E_2 \) are relatively measurable, then the difference \( E_2 \setminus E_1 \) is relatively measurable and \( |E_2 \setminus E_1|_R = |E_2|_R - |E_1|_R \). Further, the union of a finite number of disjoint relatively measurable sets \( E_1, E_2, \ldots, E_n \) is again relatively measurable and

\[
\bigcup_{j=1}^{n} E_j|_R = \sum_{j=1}^{n} |E_j|_R.
\]

We say that a system of real-valued functions \( g_1(t), g_2(t), \ldots, g_k(t) \) defined on the positive half-line is relatively measurable, if for all systems \( x_1, x_2, \ldots, x_k \) of real numbers the sets \( \bigcap_{j=1}^{k} \{t: g_j(t) < x_j\} \) are relatively measurable.

For every interval \( I = [a, b] \) and for every function \( f \) we shall use the following notation: \( f(I) = f(b) - f(a) \), \( I + t = \{u + t: u \in I\} \).

We say that a function \( f(t) \) is a relative process with independent increments, if for every positive integer \( k \) and for every system \( I_1, I_2, \ldots, I_k \) of disjoint intervals the system of functions \( g_j(t) = f(I_j + t) (j = 1, 2, \ldots, k) \) is relatively measurable,

\[
(1) \quad \bigcap_{j=1}^{k} \{t: f(I_j + t) < x_j\} |_R = \prod_{j=1}^{k} \{t: f(I_j + t) < x_j\} |_R
\]
for each $x_1, x_2, \ldots, x_k$ and

\begin{equation}
F(I, x) = |\{t: f(I+t) < x\}|_R
\end{equation}

for every interval $I$ is a probability distribution function, i.e. is a monotone non-decreasing function continuous on the left, with $F(I, -\infty) = 0$ and $F(I, \infty) = 1$. The concept of relative processes has been proposed by H. Steinhaus (see [12], [13]). It should be noted that it suffices to require condition (1) for systems of disjoint intervals $I_1, I_2, \ldots, I_k$ such that the closed intervals $\bar{I}_j$ and $\bar{I}_{j+1}$ ($j = 1, 2, \ldots, k-1$) have a common point.

The following non-effective existence theorem for relative processes with independent increments was proved in [13].

Let $f(t, \omega)$ be a measurable separable homogeneous stochastic process with independent increments. Then almost all its realizations $f(t, \omega_0)$ are relative processes with independent increments and

\begin{equation}
|\{t: f(I+t, \omega_0) < x\}|_R = \Pr\{\omega: f(I, \omega) < x\}.
\end{equation}

Some effective examples of Poisson relative processes, i.e. relative processes with independent increments having Poisson distribution were given in [12]. An example of a Gaussian relative process was given in [14]. The aim of the present paper is to give a combinatorial construction of relative processes with independent increments having continuous distribution functions (2). We shall first discuss some simple properties of distribution functions associated with a relative process, which enable us to formulate the main result of this paper. We note that a similar problem of arithmetical modelling of sequences of random variables was considered by several authors. For a complete treatment of this subject the reader is referred to the paper [10] by A. G. Postnikov, where further references to the literature can be found.

2. Distribution functions associated with relative processes. It is very easy to see that for every relative process the equation $F(I_1, x) = F(I_2, x)$ holds whenever $|I_1| = |I_2|$. This fact permits us to introduce the notation $F_{\{I\}}(x) = F(I, x)$, which is more convenient for our purpose. Thus to every relative process with independent increments there corresponds a one-parameter family $\{F_{\{I\}}\}_{t>0}$ of distribution functions completely describing relative measures (1).

**Theorem 1.** The family $\{F_{\{I\}}\}_{t>0}$ associated with a relative process with independent increments is a one-parameter semi-group under convolution, i.e. $F_{t_1} * F_{t_2} = F_{t_1 + t_2}$.

**Proof.** Let $x$ be an arbitrary continuity point of the distribution function $F_{t_1} * F_{t_2}$. For any positive number $\varepsilon$ we can find a system
$x_1 < x_2 < \ldots < x_n$ of real numbers such that

(4) \[ \sum_{j=1}^{n-1} F_{t_1}(x-x_j)(F_{t_2}(x_{j+1})-F_{t_2}(x_j)) \leq F_{t_1} * F_{t_2}(x) + \frac{\varepsilon}{3}, \]

(5) \[ \sum_{j=1}^{n-1} F_{t_1}(x-x_{j+1})(F_{t_2}(x_{j+1})-F_{t_2}(x_j)) \geq F_{t_1} * F_{t_2}(x) - \varepsilon, \]

(6) \[ F_{t_2}(x_1) \leq \frac{\varepsilon}{3} \quad \text{and} \quad 1 - F_{t_2}(x_n) \leq \frac{\varepsilon}{3}. \]

Consider the intervals $I_1 = [0, t_1)$, $I_2 = [t_1, t_1 + t_2)$ and $(I_3 = [0, t_1 + t_2)$. Put

\[ A_r(x) = \{ t : f(I_r + t) < x \} \quad (r = 1, 2, 3). \]

Of course,

(7) \[ |A_1(x)|_R = F_{t_1}(x), \quad |A_2(x)|_R = F_{t_2}(x), \quad |A_3(x)|_R = F_{t_1 + t_2}(x). \]

Since $f(I_3 + t) = f(I_1 + t) + f(I_2 + t)$, the set $A_3(x)$ is contained in the union of disjoint relatively measurable sets

\[ A_3(x) \subset A_3(x_1) \cup A_3(x_n) \cup \bigcup_{j=1}^{n-1} A_1(x-x_j) \cup (A_2(x_{j+1}) \setminus A_2(x_j)) \]

and contains the union of disjoint relatively measurable sets

\[ A_3(x) \supset \bigcup_{j=1}^{n-1} A_1(x-x_{j+1}) \cup (A_2(x_{j+1}) \setminus A_2(x_j)). \]

Hence, by virtue of (1), (2) and (7), we get the inequalities

\[ F_{t_1 + t_2}(x) \leq |A_3(x_1)|_R + |A_3'(x_n)|_R + \]

\[ + \sum_{j=1}^{n-1} |A_1(x-x_j) \cup A_2(x_{j+1})|_R - \sum_{j=1}^{n-1} |A_1(x-x_j) \cap A_2(x_j)|_R \]

\[ = F_{t_2}(x_1) + 1 - F_{t_2}(x_n) + \sum_{j=1}^{n-1} F_{t_1}(x-x_j)(F_{t_2}(x_{j+1})-F_{t_2}(x_j)), \]

\[ F_{t_1 + t_2}(x) \geq \sum_{j=1}^{n-1} |A_1(x-x_{j+1}) \cup (A_2(x_{j+1}) \setminus A_2(x_j))|_R \]

\[ = \sum_{j=1}^{n-1} F_{t_1}(x-x_{j+1})(F_{t_2}(x_{j+1})-F_{t_2}(x_j)). \]

By (4) and (6) the first inequality yields

\[ F_{t_1 + t_2}(x) \leq F_{t_1} * F_{t_2}(x) + \varepsilon \]

and, by (5), the second one yields

\[ F_{t_1 + t_2}(x) \geq F_{t_1} * F_{t_2}(x) - \varepsilon. \]
Since $\varepsilon$ can be chosen arbitrarily small, we obtain the equation $F_{t_1 + t_2}(x) = F_{t_1} \ast F_{t_2}(x)$ in all continuity points $x$ of the function $F_{t_1} \ast F_{t_2}$. Hence and from the continuity on the left of both functions $F_{t_1 + t_2}$ and $F_{t_1} \ast F_{t_2}$ we get the desired result. Theorem 1 is thus proved.

It follows from Theorem 1 that the distribution functions $F_t$ associated with a relative process with independent increments are infinitely divisible. Let $\varphi_t$ be the characteristic function of $F_t$. Then $\varphi_t(s) \neq 0$ for all positive $t$ and all $s$. Since, by an argument of Fubini's type, $F_t(x)$ is for each $x$ a Lebesgue measurable function of $t$, we have, by Theorem 21.4.1 in [6], the equality $\varphi_t(s) = (\varphi_1(s))^t$ ($t > 0$).

Now consider an arbitrary characteristic function $\varphi$ of an infinitely divisible law. By well-known theorems of Kolmogorov ([8], III, § 4) and Doob ([2], p. 61 and p. 418) there exists a measurable separable homogeneous stochastic process $f(t, \omega)$ such that the characteristic function of the increment $f(I, \omega)$ is equal to $|\varphi(s)|^t$. Thus, by the theorem quoted in Chapter I, there exists a relative process having distribution functions $F_t$ which, by (3), are probability distribution functions of corresponding increments of the stochastic process in question. This yields

**Theorem 2.** A family $\{\varphi_t\}_{t>0}$ is a family of characteristic functions of distribution functions associated with a relative process with independent increments if and only if $\varphi_t(s) = (\varphi(s))^t$, where $\varphi$ is a characteristic function of an infinitely divisible law.

We note that the expression $(\varphi(s))^t = \exp t \log \varphi(s)$ is uniquely determined by defining $\log \varphi(s)$ to be continuous and vanish at the origin.

In the sequel a semi-group of distribution functions whose characteristic functions satisfy the condition of Theorem 2 will be called admissible. From Theorem 2 and Lemma 3 in [13] (Formula (30); see also [1], Theorem 1) it follows that either all distribution functions from an admissible semi-group are continuous or all distribution functions are discontinuous. In the first case the distribution functions $F_t(x)$ are continuous as functions of two variables $x$ and $t > 0$.

**3. Admissible sequences of integers.** Let $F$ be a distribution function. By $S(F)$ we denote the support of $F$, i.e. the smallest closed subset $E$ such that $\int dF(x) = 1$. In other words, $x \in S(F)$ if and only if $F(x-h) \neq F(x+h)$, where $h$ is arbitrarily small and positive. Denoting by $\overline{E}$ the closure of a set $E$ and by $E_1 + E_2$ the set $\{x+y: x \in E_1, y \in E_2\}$ we have the formula

$$S(F_1 \ast F_2) = S(F_1) + S(F_2)$$

(see [5], p. 275). In what follows we shall use the notation

$$a(F) = \inf\{x: F(x) > 0\}, \quad b(F) = \sup\{x: F(x) < 1\}.$$
LEMMA 1. Every continuous infinitely divisible distribution function \( F \) is strictly increasing in the interval \( (a(F), b(F)) \).

Proof. The characteristic function of an infinitely divisible distribution function \( F \) is given by the Lévy-Khintchine formula

\[
\varphi(s) = \exp \left\{ i\gamma s + \int_{-\infty}^{\infty} \left( e^{ius} - 1 - \frac{ius}{1 + u^2} \right) \frac{1 + u^2}{u^2} \, dG(u) \right\},
\]

where \( \gamma \) is a real constant and \( G \) is a monotone non-decreasing bounded function with \( G(-\infty) = 0 \) (see [4], p. 76). If the distribution function \( F \) is continuous, then

\[
\int_{-1}^{1} u^{-2} dG(u) = \infty
\]

(see [13] Lemmas 2 and 3 or [1], Theorem 1).

To prove the Lemma it suffices to show that the support of \( F \) is connected. If \( G(0+) - G(0-) > 0 \), then \( F \) contains a Gaussian component and, consequently, by (8), \( S(F) \) is the whole straight line. Therefore suppose that \( G(0+) - G(0-) = 0 \). Then, by (9), we have the inequality \( G(\infty) > 0 \). Consequently, for sufficiently small positive numbers \( \varepsilon \) the integrals

\[
\int_{|u| > \varepsilon} \frac{1 + u^2}{u^2} \, dG(u)
\]

are positive. Moreover, from (9) it follows that there exists a sequence \( \varepsilon_1, \varepsilon_2, \ldots \) \( (\varepsilon_n \neq 0, \ n = 1, 2, \ldots) \) tending to 0 such that

\[
\varepsilon_n S(H_n) \quad (n = 1, 2, \ldots),
\]

where the distribution function \( H_n \) is defined by the formula

\[
H_n(x) = e_n^{-1} \int_{-\infty}^{x} \chi_n(u) \frac{1 + u^2}{u^2} \, dG(u),
\]

\( \chi_n \) is the indicator of the set \( \{u: |u| > \frac{1}{2} |\varepsilon_n| \} \) and

\[
e_n = \int_{-\infty}^{\infty} \chi_n(u) \frac{1 + u^2}{u^2} \, dG(u) > 0.
\]

Consider a compound Poisson distribution function

\[
F_n = e^{-e_n} \sum_{k=0}^{\infty} \frac{c_n^k}{k!} H_n^k \quad (n = 1, 2, \ldots),
\]
where $H^*(x) = 0$, if $x \leq 0$, $H^*(x) = 1$, if $x > 0$ and $H^{(k+1)} = H^* \ast H$.

Since

$$S(F_n) = \bigcup_{k=0}^{\infty} S(H^*_n)$$

([5], p. 277), we infer, by virtue of (8), that $S(F_n)$ is the least closed additive semi-group of real numbers containing 0 and $S(H_n)$. Hence and from (10) it follows that $S(F_n)$ contains an $|\varepsilon_n|$-net. Let $\tilde{F}_n$ be a distribution function with the characteristic function

$$\psi_n(s) = \exp \left\{ i(\gamma + \gamma_n)s + \int_{-1/2|\varepsilon_n|}^{1/2|\varepsilon_n|} \left( e^{ius} - 1 - \frac{ius}{1 + u^2} \right) \frac{1 + u^2}{u^2} dG(u) \right\},$$

where

$$\gamma_n = -\int_{|u| > 1/2|\varepsilon_n|} u^{-1} dG(u).$$

Since the characteristic function $\psi_n$ of $F_n$ is equal to

$$\exp \left\{ c_n \int_{-\infty}^{\infty} (e^{ius} - 1) dH_n(u) \right\},$$

we have, by (11), the equation $\varphi(s) = \psi_n(s) \psi_n(s)$. Thus $F = F_n \ast \tilde{F}_n$ and, consequently, by (8),

$$S(F) = S(F_n) + S(\tilde{F}_n) \quad (n = 1, 2, \ldots).$$

Since $S(F_n)$ contains an $|\varepsilon_n|$-net, the last formula implies that for any $n$ the support $S(F)$ contains an $|\varepsilon_n|$-net. Thus $S(F)$ is connected, which completes the proof.

Let $\{F_t\}_{t>0}$ be an admissible semi-group of continuous distribution functions. By Lemma 1 each function $F_t$ is strictly increasing in the interval $(a(F_t), b(F_t))$ and, consequently, has an inverse function $F_t^{-1}$ in this interval. Of course, the inverse function $F_t^{-1}$ is continuous in the open interval $(0, 1)$. Let $\omega_n$ be the modulus of continuity of the function $F_{1/n}$ on the whole real line and let $\omega'_n$ be the modulus of continuity of the function $F_{1/n}$ in the interval $[n^{-2}, 1 - n^{-2}]$ $(n = 2, 3, \ldots)$. It is obvious that we can find a sequence $r_2, r_3, \ldots$ of positive integers satisfying the condition

$$\omega_n(\omega'_n(n^{-1})) = o(n^{-1}) \quad \text{as} \quad n \to \infty.$$

Every such sequence associated with $\{F_t\}_{t>0}$ will be called admissible. It should be noted that for admissible sequences $r_2, r_3, \ldots$ by virtue of the inequality $\omega_n(\omega'_n(h)) \geq h$, the asymptotic relation

$$r_n = o(n^{-1}) \quad \text{as} \quad n \to \infty$$

holds.
As an example we shall present admissible sequences associated with semigroups of symmetric stable distributions. Consider a semi-group of distribution functions $F_t$ with characteristic functions

$$\varphi_t(s) = \exp(-t|s|^\alpha),$$

where $\alpha$ is a constant satisfying the inequality $0 < \alpha \leq 2$. Of course, for $\alpha = 2$ we have a semi-group of Gaussian distributions.

We shall prove that each sequence $r_2, r_3, \ldots$ satisfying the condition

$$\lim_{n \to \infty} r_n^{-1} n^{3+2\alpha} = 0 \quad \text{if} \quad \alpha < 2,$$

or the condition

$$\lim_{n \to \infty} r_n^{-1} n^3 < \infty \quad \text{if} \quad \alpha = 2$$

is admissible for a semi-group of symmetric stable laws with exponent $\alpha$.

It is well-known ([4], p. 183) that each stable probability distribution is absolutely continuous and its density function is bounded on the whole real line. Let $p(\alpha, x)$ be the density function of $F_1(x)$. Since, by (14), $F_t(x) = F_1(x t^{-1/\alpha})$, we have the inequality

$$\omega_n(h) = \omega_1(n^{1/\alpha} h) \leq c_1 n^{1/\alpha} h,$$

where $c_1$ is a constant. Furthermore, we have the equation for inverse functions $F_t^{-1}(x) = t^{1/\alpha} F_1^{-1}(x)$. Hence we get the formula

$$\omega_n'(h) = n^{-1/\alpha} \sup |F_t^{-1}(y_1) - F_t^{-1}(y_2)|,$$

where the supremum is extended over all $y_1, y_2$ satisfying the conditions $|y_1 - y_2| \leq h$, $n^{-2} \leq y_1, y_2 \leq 1 - n^{-2}$. Since the distribution $F_1$ is symmetric and unimodal (see [7], [16]) the above supremum is not greater than $p(\alpha, x_n)^{-1} h$, where $x_n$ is defined by the equation

$$F_1(x_n) = 1 - n^{-2}.$$

Thus

$$\omega_n'(h) \leq n^{-1/\alpha} p(\alpha, x_n)^{-1} h.$$

For $\alpha < 2$ there exists a positive constant $c_2$ such that

$$\lim_{x \to \infty} \alpha^\alpha (1 - F_1(x)) = c_2$$

(see [9], p. 201 and [4], p. 182). Moreover, from a Wintner's result ([15]; see also [11]) we obtain an asymptotic formula

$$\lim_{x \to \infty} x^{1/\alpha} p(\alpha, x) = \frac{1}{\pi} I'(1+\alpha) \sin \frac{\alpha \pi}{2}.$$
Hence and from (18) and (20) it follows that there exists a constant $c_3$ such that $p(\alpha, x_n^{-1}) \leq c_3 n^{2+2/\alpha} (\alpha < 2)$. Thus, by (17) and (19),

$$w_n(\omega'_n(h)) \leq c n^{2+2/\alpha} h \quad (\alpha < 2),$$

where $c$ is a constant. Hence it follows that a sequence $r_2, r_3, \ldots$ satisfying (15) is admissible for $\alpha < 2$.

If $\alpha = 2$, then

$$p(2, x) = \frac{1}{2 \sqrt{\pi}} \exp \left( -\frac{x^2}{4} \right)$$

and

$$\lim_{x \to \infty} x(1 - F_1(x)) \exp (x^2/4) = \pi^{-1/2}$$

(see [3], p. 131). Hence and from (18) it follows that $p(2, x_n^{-1}) \leq c_3 x_n^{-1} n^2$, where $c_3$ is a constant. Thus, by (17) and (19),

$$w_n(\omega'_n(h)) \leq c x_n^{-1} n^2 h \quad (\alpha = 2),$$

where $c$ is a constant. Since $\lim_{n \to \infty} x_n = \infty$, each sequence $r_2, r_3, \ldots$ satisfying (16) is, by the last inequality, admissible.

**Lemma 2.** Let $\{F_t\}_{t \geq 0}$ be an admissible semi-group of continuous distribution functions and let $r_2, r_3, \ldots$ be an admissible sequence associated with this semi-group. If $s_1, s_2, \ldots$ is a sequence of integers satisfying the condition

$$(21) \quad \lim_{n \to \infty} \frac{s_n}{n} = d > 0,$$

then for every real number $x$ we have the formula

$$\lim_{n \to \infty} \sum_{i=1}^{s_n} F_{i/n} \left( x - \sum_{i=1}^{s_n} F_{i/n}^{-1} \left( \frac{k_i}{r_n+1} \right) \right) (r_n+1)^{-s_n} = F_d(x),$$

where the summation $\sum$ is extended over all systems $k_1, k_2, \ldots, k_{s_n}$ of integers satisfying the condition $1 \leq k_i \leq r_n$ ($i = 1, 2, \ldots, s_n$).

**Proof.** For brevity we introduce the notation

$$(22) \quad A_n(x) = \sum_{i=1}^{s_n} F_{i/n} \left( x - \sum_{i=1}^{s_n} F_{i/n}^{-1} \left( \frac{k_i}{r_n+1} \right) \right) (r_n+1)^{-s_n}.$$

Let $p_n$ and $q_n$ be integers satisfying the conditions $p_n \geq 1$, $q_n \leq r_n$,

$$(23) \quad \frac{p_n - 1}{r_n + 1} \leq \frac{1}{n^2} \leq \frac{p_n}{r_n + 1}, \quad \frac{q_n + 1}{r_n + 1} \leq 1 - \frac{1}{n^2} < \frac{q_n + 2}{r_n + 1}.$$
Put
\[ B_n(x) = \sum_s F_{1/n} \left( x - \sum_{i=1}^{s_n} F^{-1}_{1/n} \left( \frac{k_i}{r_n+1} \right) \right) (r_n+1)^{-s_n}, \]
where the summation \( \sum \) is extended over all systems \( k_1, k_2, \ldots, k_{s_n} \) of integers satisfying the condition \( p_n \leq k_i \leq q_n \) (\( i = 1, 2, \ldots, s_n \)). By a simple reasoning we obtain the inequality
\[ |A_n(x) - B_n(x)| \leq \sum_{j=1}^{s_n} \sum_{(j)} F_{1/n} \left( x - \sum_{i=1}^{s_n} F^{-1}_{1/n} \left( \frac{k_i}{r_n+1} \right) \right) (r_n+1)^{-s_n}, \]
where the summation \( \sum_{(j)} \) is running over all systems \( k_1, k_2, \ldots, k_{s_n} \) of integers satisfying the conditions \( 1 \leq k_i \leq r_n \) (\( i = 1, 2, \ldots, s_n \)) and \( k_j \neq p_n, p_n+1, \ldots, q_n-1, q_n \). Hence we get the inequality
\[ |A_n(x) - B_n(x)| \leq s_n (p_n-1+r_n-q_n) r_n^{-s_n}(r_n+1)^{-s_n} \leq s_n (p_n+r_n-q_n) (r_n+1)^{-1}. \]

Finally, taking into account (13), (21) and (23), we obtain the formula
\[ \lim_{n \to \infty} (A_n(x) - B_n(x)) = 0. \]

Consider the expression
\[ C_n(x) = \sum_{a_{k_1+1} \ a_{k_2+1} \ \ldots \ \ a_{k_{s_n}+1}} \int \int \ldots \int F_{1/n} \left( x - \sum_{i=1}^{s_n} a_{k_i} \right) dF_{1/n}(x_1) dF_{1/n}(x_2) \ldots dF_{1/n}(x_{s_n}), \]
where
\[ a_{k_i} = F^{-1}_{1/n} \left( \frac{k_i}{r_n+1} \right). \]

Since
\[ \int_{a_{k_i}}^{a_{k_i+1}} dF_{1/n}(x_i) = \frac{k_i+1}{r_n+1} - \frac{k_i}{r_n+1} = \frac{1}{r_n+1}, \]
the expression (24) can be written in the form
\[ B_n(x) = \sum_{a_{k_1+1} \ a_{k_2+1} \ \ldots \ \ a_{k_{s_n}+1}} \int \int \ldots \int F_{1/n} \left( x - \sum_{i=1}^{s_n} a_{k_i} \right) dF_{1/n}(x_1) dF_{1/n}(x_2) \ldots dF_{1/n}(x_{s_n}). \]
Thus
\begin{equation}
(B_n(x) - C_n(x)) \leq \sum \omega_n \left( \sum_{i=1}^{s_n} |a_{k_{i+1}} - a_{k_i}| \right) \int F_{1/n}^{a_{k_{i+1}}} \left( \frac{k_i + 1}{r_n + 1} \right) \leq \omega_n'(r_n^{-1}),
\end{equation}

whenever \( p_n \leq k_i \leq q_n \). Thus, by (21) and by well-known formula \( \omega_n'(m\hbar) \leq m \omega_n'(\hbar) \) \( m = 1, 2, \ldots \), inequality (28) implies
\begin{equation}
(B_n(x) - C_n(x)) \leq s_n \omega_n(\omega_n'(r_n^{-1})) = d_n \omega_n(\omega_n'(r_n^{-1})) + o(1),
\end{equation}
which, by (12), yields
\begin{equation}
\lim_{n \to \infty} (B_n(x) - C_n(x)) = 0.
\end{equation}

Further, from (13), (21), (23), (26) and from the formula
\begin{equation}
F_{(e_{n+1})/n}(x) = F_{1/n}^{a_{e_{n+1}}} = \int \int \ldots \int F_{1/n}^{a_{e_{n+1}}} (x - \sum_{i=1}^{s_n} x_i) dF_{1/n}(x_1) dF_{1/n}(x_2) \ldots dF_{1/n}(x_{s_n})
\end{equation}

with (25) and (29) implies the assertion of the Lemma.

4. A combinatorial construction of relative processes. In this Chapter we shall give an effective combinatorial construction of relative processes with independent increments having continuous distribution functions.

**Theorem 3.** Let \( \{F_i\}_{i=0} \) be an admissible semi-group of continuous distribution functions and let \( r_2, r_3, \ldots \) be an admissible sequence of integers associated with this semi-group. For every \( n \geq 2 \) let \( \langle k_{1/2}^{(n)}, k_{2/3}^{(n)}, \ldots, k_{r_{n-1}}^{(n)} \rangle \).
j = 1, 2, ..., r_n^m, be a sequence of all ordered r_n-tuples of positive integers not exceeding r_n. Put a_n = r_n^m, b_n = \sum_{s=1}^{n} r_s^{1+r_r} r_{s+1}^{r_{s+1}+r_{s+1}+1} (n \geq 2), b_1 = 0, H(t) = 0, if t < 0 and H(t) = 1 if t \geq 0. Then the function

\[ f(t) = \sum_{n=2}^{\infty} \sum_{i=1}^{r_n} \sum_{j=1}^{u_n} \sum_{m=1}^{n r_n^{1+a_n^m+1}} F_{i|j+m}(\frac{b_{n+1}^m}{r_n+1}) \times 
\times H\left(t - b_{n+1}^m - \frac{(n-1)r_n a_n + (j-1)r_n + (i-1)}{n}\right) \]

is a relative process with independent increments. Moreover, \{F_t\}_{t \geq 0} is the family of its distribution functions.

Proof. Consider a system of intervals \( I_p = [c_{p-1}, c_p) (p = 1, 2, ..., k) \), where \( c_0 = 0 \). In what follows we assume that the index \( n \) satisfies the conditions \( n \geq 2 \) and \( \min_{1 \leq p < k} |I_p| > 2n^{-1} \). For every such index \( n \) we can define an auxiliary system of intervals

\[ I_{pn} = \left[ \frac{u_{pn}}{n}, \frac{v_{pn}}{n} \right] (p = 1, 2, ..., k) \]

where \( u_{pn}, v_{pn} \) are integers, 

(30) \hspace{1cm} u_{1n} = 0, \hspace{1cm} nc_{p-1} \leq u_{pn} \leq nc_{p-1} + 1, \hspace{1cm} nc_{p-1} - 1 \leq v_{pn} \leq nc_{p} \hspace{1cm} (p = 1, 2, ..., k) \)

and

(31) \hspace{1cm} u_{p+1,n} = v_{pn} + 1 \hspace{1cm} (p = 1, 2, ..., k-1).

Of course, \( I_{pn} \subset I_p \hspace{1cm} (p = 1, 2, ..., k) \) and

(32) \hspace{1cm} \lim_{n \to \infty} |I_{pn}| = |I_p| \hspace{1cm} (p = 1, 2, ..., k).

Moreover, by (31), the distance between two consecutive intervals \( I_{pn} \) and \( I_{p+1,n} \) is equal to \( n^{-1} \).

Let us introduce the notation

\[ U(n, m) = \left[ b_{n-1} + \frac{(m-1)r_n a_n}{n}, b_{n-1} + \frac{nr_n a_n}{n} \right), \]

where \( m = 1, 2, ..., nr_{n+1} a_{n+1} \) and \( n = 2, 3, ... \). Further, for any system \( y_1, y_2, ..., y_k \) of real numbers we put

(33) \hspace{1cm} A(n, m; y_1, y_2, ..., y_k) = \bigcap_{p=1}^{k} \{t: I_{pn} + t \subset U(n, m), f(I_{pn} + t) \leq y_p\}.

By the definition of the function \( f \) the distance between its consecutive jump points in the interval \( U(n, m) \) is equal to \( n^{-1} \). Put

(34) \hspace{1cm} w_{pn} = v_{pn} - u_{pn} \hspace{1cm} (p = 1, 2, ..., k).
If $I_{pm} + t_0$ is contained in $U(n, m)$, then the interval $I_{pm} + t_0$ contains exactly $w_{pn}$ jump points of the function $f$. Furthermore, the same jump points belong to every interval $I_{pm} + t$, where $t$ is taken from an interval of the length $n^{-1}$ containing $t_0$. Thus as $n \to \infty$ we have

$$A(n, m; y_1, y_2, \ldots, y_k) = n^{-1}a(n, m; y_1, y_2, \ldots, y_k) + O(n^{-1})$$

uniformly in $m$, where $a(n, m; y_1, y_2, \ldots, y_k)$ is the number of all $\sum_{p=1}^{k} w_{pn}$-tuples of consecutive jump points in the interval $U(n, m)$ such that the sum of $w_{pn}$ first jumps is less or equal to $y_1$, the sum of next $w_{2n}$ jumps is less or equal to $y_2$ and so on.

Now we shall establish an asymptotic formula for $a(n, m; y_1, y_2, \ldots, y_k)$. The jump points of the function $f$ in the interval $U(n, m)$ are of the form

$$b_{n-1} + \frac{(m-1)r_n a_n + (j-1)r_n + (i-1)}{n} \quad (i = 1, 2, \ldots, r_n; j = 1, 2, \ldots, a_n).$$

We note that the number of $\sum_{p=1}^{k} w_{pn}$-tuples of consecutive jump points in $U(n, m)$ containing at least two jump points with different indices $j$ is not greater than $a_n \sum_{p=1}^{k} w_{pn}$, which is of order $o(r_n a_n)$ uniformly in $m$ as $n \to \infty$. Consequently, the number $a(n, m; y_1, y_2, \ldots, y_k)$ is equal, with an accuracy $o(r_n a_n)$, to the number of all $\sum_{p=1}^{k} w_{pn}$-tuples of consecutive jump points in $U(n, m)$ corresponding to the same index $j$ and satisfying the requirements formulated in the definition of $a(n, m; y_1, y_2, \ldots, y_k)$. In other words, the number $a(n, m; y_1, y_2, \ldots, y_k)$ is equal, with an accuracy $o(r_n a_n)$, to the number of all pairs $\langle j, s \rangle$ $(j = 1, 2, \ldots, a_n; s = 0, 1, \ldots, r_n - \sum_{p=1}^{k} w_{pn})$ for which the following inequalities are true:

$$\sum_{i=\sigma_{p-1,n}+1}^{\sigma_{p,n}} F_{1/n}^{-1}\left(\frac{k_{i+s,j}^{(n)}}{r_{n} + 1}\right) \leq y_p \quad (p = 1, 2, \ldots, k),$$

where

$$\sigma_{0,n} = 0, \quad \sigma_{p,n} = \sum_{q=1}^{p} w_{qn} \quad (p = 1, 2, \ldots, k).$$

Further, the last inequalities are equivalent to the following ones:

$$k_{\sigma_{p-1,n}+1+s,j}^{(n)} \leq (r_n + 1) F_{1/n}\left(y_p - \sum_{i=\sigma_{p-1,n}+2}^{\sigma_{p,n}} F_{1/n}^{-1}\left(\frac{k_{i+s,j}^{(n)}}{r_{n} + 1}\right)\right) \quad (p = 1, 2, \ldots, k).$$
From the definition of \( r_n \)-tuples \( \langle k_{i_1}^{(n)}, k_{i_2}^{(n)}, \ldots, k_{i_{n+1}}^{(n)} \rangle \), by a combinatorial argument, it follows that for any fixed index \( s \) the number of indices \( j \) satisfying (36) is given by the formula
\[
\sum r_n^{r_n - z_{kn}} \prod_{p=1}^k \left( (r_n + 1)F_{1/n}(y_p - \sum_{i=2}^{s_{pn}} F_{1/n}^{-1} \left( \frac{d_{pi}}{r_n + 1} \right)) \right),
\]
where the summation is extended over all systems \( d_{pi} \) \( (i = z_{p-1,n+2}, \ldots, z_{pn}; p = 1, 2, \ldots, k) \) of integers satisfying the condition \( 1 \leq d_{pi} \leq r_n \) and \([x]\) denotes the integral part of \( x \). Setting
\[
(s_{pn} = z_{pn} - z_{p-1,n+1} = w_{pn} - 1 \quad (p = 1, 2, \ldots, k),
\]
we can write the last expression in the form
\[
a_n \prod_{p=1}^k \left( \sum_{(p)} F_{1/n}(y_p - \sum_{i=1}^{s_{pn}} F_{1/n}^{-1} \left( \frac{k_{pi}}{r_n + 1} \right)) (r_n + 1)^{-s_{pn}} \right) + o(a_n),
\]
where the summation \( \sum_{(p)} \) is extended over all systems \( k_{p1}, k_{p2}, \ldots, k_{nsp_n} \) of integers satisfying the condition \( 1 \leq k_{pi} \leq r_n \) \( (i = 1, 2, \ldots, s_{pn}) \). Moreover, since this expression does not depend on the index \( s \) and \( 0 \leq s \leq z_{kn} \), the number of pairs \( \langle j, s \rangle \) satisfying (36) and, consequently, the number \( a(n, m; y_1, y_2, \ldots, y_k) \) are given by the formula
\[
a_n (r_n - z_{kn}) \prod_{p=1}^k \left( \sum_{(p)} F_{1/n}(y_p - \sum_{i=1}^{s_{pn}} F_{1/n}^{-1} \left( \frac{k_{pi}}{r_n + 1} \right)) (r_n + 1)^{-s_{pn}} \right) + o(r_n a_n).
\]

From (32), (34) and (37) it follows that \( \lim_{n \to \infty} s_{pn}/n = |I_p| \) and \( z_{kn} = O(n) \), which, by (13), implies \( z_{kn} = o(r_n) \). Thus, by Lemma 2 and formula (35),
\[
A(n, m; y_1, y_2, \ldots, y_k) = n^{-1}r_n a_n \prod_{p=1}^k F_{1/n}(y_p) + o(n^{-1}r_n a_n)
\]
uniformly in \( m \).

Now consider the set
\[
B_p(n, m; \varepsilon) = \{ t; I_p + t \subset U(n, m), |f(I_p + t) - f(I_{pn} + t)| > \varepsilon \},
\]
where \( \varepsilon \) is a positive number. By (30) we conclude that if the interval \( I_p + t \) is contained in \( U(n, m) \), then the set \( (I_p + t) \setminus (I_{pn} + t) \) contains at most two jump points of the function \( f \). Thus the function \( f \) has a saltus of absolute magnitude greater than \( \frac{1}{2} \varepsilon \) in the set \( (I_p + t) \setminus (I_{pn} + t) \).
whenever \( t \in B_p(n, m; \varepsilon) \). Since, by (30)

\[
| (I_p + t) \setminus (I_{pn} + t) | \leq 2n^{-1},
\]

we have the inequality

\[(40)\]

\[
| B_p(n, m; \varepsilon) | \leq 2n^{-1} b_p(n, m; \varepsilon),
\]

where \( b_p(n, m; \varepsilon) \) is the number of jumps of the function \( f \) in \( U(n, m) \) of absolute magnitude greater than \( \frac{1}{2} \varepsilon \). In other words, \( b_p(n, m; \varepsilon) \) is equal to the number of integers \( k_{ij}^{(n)} (i = 1, 2, \ldots, r_n; j = 1, 2, \ldots, a_n) \) for which

\[
\left| \frac{k_{ij}^{(n)}}{r_n + 1} \right| > \frac{1}{2} \varepsilon.
\]

Since the last inequality is equivalent to the union of two inequalities

\[
k_{ij}^{(n)} < F_{1/n}(-\frac{1}{2} \varepsilon)(r_n + 1), \quad k_{ij}^{(n)} > F_{1/n}(\frac{1}{2} \varepsilon)(r_n + 1),
\]

we obtain by a simple combinatorial reasoning an estimation

\[
b_p(n, m; \varepsilon) \leq a_n(r_n + 1) \left( 1 - F_{1/n}(\frac{1}{2} \varepsilon) + F_{1/n}(-\frac{1}{2} \varepsilon) \right).
\]

Hence, by (40), we get the inequality

\[(41)\]

\[
| B_p(n, m; \varepsilon) | \leq 2n^{-1} a_n(r_n + 1) \left( 1 - F_{1/n}(\frac{1}{2} \varepsilon) + F_{1/n}(-\frac{1}{2} \varepsilon) \right).
\]

Further, setting

\[(42)\]

\[
C_p(n, m) = \{ t : I_{pn} + t \subset U(n, m), I_p + t \notin U(n, m) \} \cup
\]

\[
\cup \{ t : I_p + t \subset U(n, m) \} \setminus U(n, m) \cup \{ t : I_p + t \in U(n, m) \}.
\]

we have the inequality

\[(43)\]

\[
| C_p(n, m)| \leq 2n^{-1} + 2 | I_p |.
\]

For every positive number \( \varepsilon \), taking into account (33), (39) and (42), we obtain the inclusions

\[
U(n, m) \setminus \bigcap_{p=1}^k \{ t : f(I_p + t) < x_p \} \subset A(n, m; x_1 + \varepsilon, x_2 + \varepsilon, \ldots, x_k + \varepsilon)
\]

\[
\cup \bigcup_{p=1}^k B_p(n, m; \varepsilon) \cup \bigcup_{p=1}^k C_p(n, m),
\]

\[
A(n, m; x_1 - \varepsilon, x_2 - \varepsilon, \ldots, x_k - \varepsilon) \subset U(n, m) \setminus \bigcap_{p=1}^k \{ t : f(I_p + t) < x_p \} \cup
\]

\[
\cup \bigcup_{p=1}^k B_p(n, m; \varepsilon) \cup \bigcup_{p=1}^k C_p(n, m).
\]
Hence and from (38), (41) and (43) we get the inequalities

\begin{align*}
U(n, m) \cap \bigcap_{p=1}^{k} \{ t : f(I_p + t) < x_p \} & \leq n^{-1}r_n a_n \prod_{p=1}^{k} F_{|I_p|}(x_p + \varepsilon) + \\
& + 2kn^{-1}r_n a_n \left( 1 - F_{1/n}(1/2\varepsilon) + F_{1/n}(-1/2\varepsilon) \right) + o(n^{-1}r_n a_n), \\
U(n, m) \cap \bigcap_{p=1}^{k} \{ t : f(I_p + t) < x_p \} & \geq n^{-1}r_n a_n \prod_{p=1}^{k} F_{|I_p|}(x_p - \varepsilon) + \\
& + 2kn^{-1}r_n a_n \left( F_{1/n}(1/2\varepsilon) - 1 - F_{1/n}(-1/2\varepsilon) \right) + o(n^{-1}r_n a_n)
\end{align*}

uniformly in \( n \).

By the definition of numbers \( a_n \) and \( b_n \) for every positive number \( T \) there exist integers \( N \) and \( M \) satisfying the conditions

\[ b_{N-1} + \frac{Mr_N a_N}{N} \leq T < b_{N-1} + \frac{(M+1)r_N a_N}{N}, \quad 1 \leq M \leq N r_{N+1} a_{N+1}. \]

Since \( b_{N-1} \geq r_N a_N r_{N-1} a_{N-1} \), we have \( N^{-1}r_N a_N = o(b_{N-1}) \) and consequently, \( N^{-1}r_N a_N = o(T) \). Thus

\[ T = b_{N-1} + \frac{Mr_N a_N}{N} + o(T). \]

Further, taking into account the decomposition

\[ \left( 0, b_{N-1} + \frac{Mr_N a_N}{N} \right) = \bigcup_{n=2}^{N-1} \bigcup_{m=1}^{n} U(n, m) \cup \bigcup_{m=1}^{M} U(N, m), \]

the formula \( |U(n, m)| = n^{-1}r_n a_n \) and the limit relation for \( \varepsilon > 0 \),

\[ \lim_{n \to \infty} \left( 1 - F_{1/n}(1/2\varepsilon) + F_{1/n}(-1/2\varepsilon) \right) = 0, \]

we obtain, by (44) and (45), the inequalities

\[ \bigcap_{p=1}^{k} \{ t : f(I_p + t) < x_p \} \cap [0, T] \leq T \prod_{p=1}^{k} F_{|I_p|}(x_p + \varepsilon) + o(T), \]

\[ \bigcap_{p=1}^{k} \{ t : f(I_p + t) < x_p \} \cap [0, T] \geq T \prod_{p=1}^{k} F_{|I_p|}(x_p - \varepsilon) + o(T). \]

Hence we get the formulas

\[ \bigcap_{p=1}^{k} \{ t : f(I_p + t) < x_p \} \wedge R \leq \prod_{p=1}^{k} F_{|I_p|}(x_p + \varepsilon), \]

\[ \bigcap_{p=1}^{k} \{ t : f(I_p + t) < x_p \} \wedge R \geq \prod_{p=1}^{k} F_{|I_p|}(x_p - \varepsilon), \]
which, by virtue of the arbitrariness of $\varepsilon$ and the continuity of distribution functions $F_{|I_p|} (p = 1, 2, \ldots, k)$, imply the equality

$$\left| \bigcap_{p=1}^{k} \{ t : f(\xi_p + t) < x_p \} \right|_R = \prod_{p=1}^{k} F_{|I_p|}(x_p).$$

Thus the function $f$ is a relative process with distribution functions \{$F_t$\}_{t>0}, which completes the proof.

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