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RELATIVE PROCESSES WITH CONTINUOUS DISTRIBUTION FUNCTIONS

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1. Introduction. Let E be a Lebesgue measurable subset of the positive half-line. By $|E|$ we shall denote the Lebesgue measure of E. The limits

$$|E|_{\underline{R}} = \varliminf_{T\to\infty} \frac{1}{T} |E\cap[0, T)|, \qquad |E|_{\overline{R}} = \varlimsup_{T\to\infty} \frac{1}{T} |E\cap[0, T)|$$

are called *the lower relative measure* of E and *the upper relative measure* of E respectively. If $|E|_{\underline{R}} = |E|_{\overline{R}}$, the set E is said to be *relatively measurable*; its lower and upper relative measures are then called simply *relative measures* and denoted by $|E|_R$. Obviously, the complement E' of a relatively measurable set E is also relatively measurable and $|E'|_R = 1-|E|_R$. Moreover, if $E_1 \subset E_2$ and both E_1 and E_2 are relatively measurable, then the difference $E_2 \setminus E_1$ is relatively measurable and $|E_2\setminus E_1|_R = |E_2|_R - |E_1|_R$. Further, the union of a finite number of disjoint relatively measurable sets $E_1, E_2, \ldots, E_n$ is again relatively measurable and $|\bigcup_{j=1}^{n} E_j|_R = \sum_{j=1}^{n} |E_j|_R$.

We say that a system of real-valued functions $g_1(t), g_2(t), \ldots, g_k(t)$ defined on the positive half-line is *relatively measurable*, if for all systems $x_1, x_2, \ldots, x_k$ of real numbers the sets $\bigcap_{j=1}^{k} \{t\colon g_j(t) < x_j\}$ are relatively measurable.

For every interval $I = [a, b]$ and for every function f we shall use the following notation: $f(I) = f(b)-f(a)$, $I+t = \{u+t\colon u\in I\}$.

We say that a function $f(t)$ is *a relative process with independent increments*, if for every positive integer k and for every system $I_1, I_2, \ldots, I_k$ of disjoint intervals the system of functions $g_j(t) = f(I_j+t)$ $(j = 1, 2, \ldots, k)$ is relatively measurable,

$$(1) \qquad \Big|\bigcap_{j=1}^{k} \{t\colon f(I_j+t) < x_j\}\Big|_R = \prod_{j=1}^{k} |\{t\colon f(I_j+t) < x_j\}|_R$$

for each $x_1, x_2, \ldots, x_k$ and

$$F(I, x) = |\{t: f(I+t) < x\}|_R \tag{2}$$

for every interval I is a probability distribution function, i.e. is a monotone non-decreasing function continuous on the left, with $F(I, -\infty) = 0$ and $F(I, \infty) = 1$. The concept of relative processes has been proposed by H. Steinhaus (see [12], [13]). It should be noted that it suffices to require condition (1) for systems of disjoint intervals $I_1, I_2, \ldots, I_k$ such that the closed intervals $\bar{I}_j$ and $\bar{I}_{j+1}$ $(j = 1, 2, \ldots, k-1)$ have a common point.

The following non-effective existence theorem for relative processes with independent increments was proved in [13].

Let $f(t, \omega)$ be a measurable separable homogeneous stochastic process with independent increments. Then almost all its realizations $f(t, \omega_0)$ are relative processes with independent increments and

$$|\{t: f(I+t, \omega_0) < x\}|_R = \Pr\{\omega: f(I, \omega) < x\}. \tag{3}$$

Some effective examples of Poisson relative processes, i.e. relative processes with independent increments having Poisson distribution were given in [12]. An example of a Gaussian relative process was given in [14]. The aim of the present paper is to give a combinatorial construction of relative processes with independent increments having continuous distribution functions (2). We shall first discuss some simple properties of distribution functions associated with a relative process, which enable us to formulate the main result of this paper. We note that a similar problem of arithmetical modelling of sequences of random variables was considered by several authors. For a complete treatment of this subject the reader is referred to the paper [10] by A. G. Postnikov, where further references to the literature can be found.

2. Distribution functions associated with relative processes. It is very easy to see that for every relative process the equation $F(I_1, x) = F(I_2, x)$ holds whenever $|I_1| = |I_2|$. This fact permits us to introduce the notation $F_{|I|}(x) = F(I, x)$, which is more convenient for our purpose. Thus to every relative process with independent increments there corresponds a one-parameter family $\{F_t\}_{t>0}$ of distribution functions completely describing relative measures (1).

THEOREM 1. *The family $\{F_t\}_{t>0}$ associated with a relative process with independent increments is a one-parameter semi-group under convolution, i.e. $F_{t_1} * F_{t_2} = F_{t_1+t_2}$.*

Proof. Let x be an arbitrary continuity point of the distribution function $F_{t_1} * F_{t_2}$. For any positive number ε we can find a system

$x_1 < x_2 < \ldots < x_n$ of real numbers such that

$$\sum_{j=1}^{n-1} F_{t_1}(x-x_j)\big(F_{t_2}(x_{j+1})-F_{t_2}(x_j)\big) \leqslant F_{t_1} * F_{t_2}(x)+\frac{\varepsilon}{3}, \tag{4}$$

$$\sum_{j=1}^{n-1} F_{t_1}(x-x_{j+1})\big(F_{t_2}(x_{j+1})-F_{t_2}(x_j)\big) \geqslant F_{t_1} * F_{t_2}(x)-\varepsilon, \tag{5}$$

$$F_{t_2}(x_1) \leqslant \frac{\varepsilon}{3} \quad \text{and} \quad 1-F_{t_2}(x_n) \leqslant \frac{\varepsilon}{3}. \tag{6}$$

Consider the intervals $I_1 = [0, t_1)$, $I_2 = [t_1, t_1+t_2)$ and $(I_3 = [0, t_1+t_2)$. Put

$$A_r(x) = \{t\colon f(I_r+t) < x\} \quad (r = 1, 2, 3).$$

Of course,

$$|A_1(x)|_R = F_{t_1}(x), \quad |A_2(x)|_R = F_{t_2}(x), \quad |A_3(x)|_R = F_{t_1+t_2}(x). \tag{7}$$

Since $f(I_3+t) = f(I_1+t)+f(I_2+t)$, the set $A_3(x)$ is contained in the union of disjoint relatively measurable sets

$$A_3(x) \subset A_2(x_1) \cup A_2'(x_n) \cup \bigcup_{j=1}^{n-1} A_1(x-x_j) \cap \big(A_2(x_{j+1}) \setminus A_2(x_j)\big)$$

and contains the union of disjoint relatively measurable sets

$$A_3(x) \supset \bigcup_{j=1}^{n-1} A_1(x-x_{j+1}) \cap \big(A_2(x_{j+1}) \setminus A_2(x_j)\big).$$

Hence, by virtue of (1), (2) and (7), we get the inequalities

$$\begin{aligned} F_{t_1+t_2}(x) \leqslant\; & |A_2(x_1)|_R + |A_2'(x_n)|_R + \\ & + \sum_{j=1}^{n-1} |A_1(x-x_j) \cap A_2(x_{j+1})|_R - \sum_{j=1}^{n-1} |A_1(x-x_j) \cap A_2(x_j)|_R \\ & = F_{t_2}(x_1)+1-F_{t_2}(x_n)+\sum_{j=1}^{n-1} F_{t_1}(x-x_j)\big(F_{t_2}(x_{j+1})-F_{t_2}(x_j)\big), \end{aligned}$$

$$\begin{aligned} F_{t_1+t_2}(x) &\geqslant \sum_{j=1}^{n-1} |A_1(x-x_{j+1}) \cap \big(A_2(x_{j+1}) \setminus A_2(x_j)\big)|_R \\ &= \sum_{j=1}^{n-1} F_{t_1}(x-x_{j+1})\big(F_{t_2}(x_{j+1})-F_{t_2}(x_j)\big). \end{aligned}$$

By (4) and (6) the first inequality yields

$$F_{t_1+t_2}(x) \leqslant F_{t_1} * F_{t_2}(x)+\varepsilon$$

and, by (5), the second one yields

$$F_{t_1+t_2}(x) \geqslant F_{t_1} * F_{t_2}(x)-\varepsilon.$$

Since ε can be chosen arbitrarily small, we obtain the equation $F_{t_1+t_2}(x) = F_{t_1} * F_{t_2}(x)$ in all continuity points x of the function $F_{t_1} * F_{t_2}$. Hence and from the continuity on the left of both functions $F_{t_1+t_2}$ and $F_{t_1} * F_{t_2}$ we get the desired result. Theorem 1 is thus proved.

It follows from Theorem 1 that the distribution functions F_t associated with a relative process with independent increments are infinitely divisible. Let φ_t be the characteristic function of F_t. Then $\varphi_t(s) \neq 0$ for all positive t and all s. Since, by an argument of Fubini's type, $F_t(x)$ is for each x a Lebesgue measurable function of t, we have, by Theorem 21. 4. 1 in [6], the equality $\varphi_t(s) = \big(\varphi_1(s)\big)^t$ $(t > 0)$.

Now consider an arbitrary characteristic function φ of an infinitely divisible law. By well-known theorems of Kolmogorov ([8], III, § 4) and Doob ([2], p. 61 and p. 418) there exists a measurable separable homogeneous stochastic process $f(t, \omega)$ such that the characteristic function of the increment $f(I, \omega)$ is equal to $\big(\varphi(s)\big)^{|I|}$. Thus, by the theorem quoted in Chapter I, there exists a relative process having distribution functions F_t which, by (3), are probability distribution functions of corresponding increments of the stochastic process in question. This yields

THEOREM 2. *A family $\{\varphi_t\}_{t>0}$ is a family of characteristic functions of distribution functions associated with a relative process with independent increments if and only if $\varphi_t(s) = \big(\varphi(s)\big)^t$, where φ is a characteristic function of an infinitely divisible law.*

We note that the expression $\big(\varphi(s)\big)^t = \exp t \log\varphi(s)$ is uniquely determined by defining $\log\varphi(s)$ to be continuous and vanish at the origin.

In the sequel a semi-group of distribution functions whose characteristic functions satisfy the condition of Theorem 2 will be called *admissible*. From Theorem 2 and Lemma 3 in [13] (Formula (30); see also [1], Theorem 1) it follows that either all distribution functions from an admissible semi-group are continuous or all distribution functions are discontinous. In the first case the distribution functions $F_t(x)$ are continuous as functions of two variables x and $t > 0$.

3. Admissible sequences of integers. Let F be a distribution function. By $S(F)$ we denote the support of F, i.e. the smallest closed subset E such that $\int_E dF(x) = 1$. In other words, $x \in S(F)$ if and only if $F(x-h) \neq F(x+h)$, where h is arbitrarily small and positive. Denoting by $\bar{E}$ the closure of a set E and by E_1+E_2 the set $\{x+y \colon x \in E_1, y \in E_2\}$ we have the formula

$$S(F_1 * F_2) = \overline{S(F_1)+S(F_2)} \tag{8}$$

(see [5], p. 275). In what follows we shall use the notation

$$a(F) = \inf\{x \colon F(x) > 0\}, \quad b(F) = \sup\{x \colon F(x) < 1\}.$$

LEMMA 1. *Every continuous infinitely divisible distribution function F is strictly increasing in the interval $\big(a(F), b(F)\big)$.*

Proof. The characteristic function of an infinitely divisible distribution function F is given by the Lévy-Khintchine formula

$$\varphi(s) = \exp\left\{i\gamma s + \int_{-\infty}^{\infty}\left(e^{ius}-1-\frac{ius}{1+u^2}\right)\frac{1+u^2}{u^2}\,dG(u)\right\},$$

where γ is a real constant and G is a monotone non-decreasing bounded function with $G(-\infty)=0$ (see [4], p. 76). If the distribution function F is continuous, then

$$\int_{-1}^{1} u^{-2}dG(u) = \infty \tag{9}$$

(see [13] Lemmas 2 and 3 or [1], Theorem 1).

To prove the Lemma it suffices to show that the support of F is connected. If $G(0+)-G(0-)>0$, then F contains a Gaussian component and, consequently, by (8), $S(F)$ is the whole straight line. Therefore suppose that $G(0+)-G(0-)=0$. Then, by (9), we have the inequality $G(\infty)>0$. Consequently, for sufficiently small positive numbers ε the integrals

$$\int_{|u|>\varepsilon}\frac{1+u^2}{u^2}\,dG(u)$$

are positive. Moreover, from (9) it follows that there exists a sequence $\varepsilon_1, \varepsilon_2, \ldots$ $(\varepsilon_n \neq 0,\ n=1,2,\ldots)$ tending to 0 such that

$$\varepsilon_n \in S(H_n) \quad (n=1,2,\ldots), \tag{10}$$

where the distribution function H_n is defined by the formula

$$H_n(x) = c_n^{-1}\int_{-\infty}^{x}\chi_n(u)\,\frac{1+u^2}{u^2}\,dG(u), \tag{11}$$

χ_n is the indicator of the set $\{u\colon |u|>\frac{1}{2}|\varepsilon_n|\}$ and

$$c_n = \int_{-\infty}^{\infty}\chi_n(u)\frac{1+u^2}{u^2}\,dG(u) > 0.$$

Consider a compound Poisson distribution function

$$F_n = e^{-c_n}\sum_{k=0}^{\infty}\frac{c_n^k}{k!}H_n^{*k} \quad (n=1,2,\ldots),$$

where $H_n^{*0}(x) = 0$, if $x \leqslant 0$, $H_n^{*0}(x) = 1$, if $x > 0$ and $H_n^{*(k+1)} = H_n^{*k} * H_n$
Since

$$S(F_n) = \overline{\bigcup_{k=0}^{\infty} S(H_n^{*k})}$$

([5], p. 277), we infer, by virtue of (8), that $S(F_n)$ is the least closed additive semi-group of real numbers containing 0 and $S(H_n)$. Hence and from (10) it follows that $S(F_n)$ contains an $|\varepsilon_n|$-net. Let $\tilde{F}_n$ be a distribution function with the characteristic function

$$\psi_n(s) = \exp\left\{i(\gamma+\gamma_n)s + \int_{-1/2|\varepsilon_n|}^{1/2|\varepsilon_n|} \left(e^{ius} - 1 - \frac{ius}{1+u^2}\right)\frac{1+u^2}{u^2}\, dG(u)\right\},$$

where

$$\gamma_n = - \int_{|u|>1/2|\varepsilon_n|} u^{-1} dG(u).$$

Since the characteristic function φ_n of F_n is equal to

$$\exp\left(c_n \int_{-\infty}^{\infty} (e^{ius} - 1)\, dH_n(u)\right),$$

we have, by (11), the equation $\varphi(s) = \varphi_n(s)\psi_n(s)$. Thus $F = F_n * \tilde{F}_n$ and, consequently, by (8),

$$S(F) = \overline{S(F_n) + S(\tilde{F}_n)} \quad (n = 1, 2, \ldots).$$

Since $S(F_n)$ contains an $|\varepsilon_n|$-net, the last formula implies that for any n the support $S(F)$ contains an $|\varepsilon_n|$-net. Thus $S(F)$ is connected, which completes the proof.

Let $\{F_t\}_{t>0}$ be an admissible semi-group of continuous distribution functions. By Lemma 1 each function F_t is strictly increasing in the interval $\big(a(F_t), b(F_t)\big)$ and, consequently, has an inverse function F_t^{-1} in this interval. Of course, the inverse function F_t^{-1} is continuous in the open interval $(0, 1)$. Let ω_n be the modulus of continuity of the function $F_{1/n}$ on the whole real line and let ω_n' be the modulus of continuity of the function $F_{1/n}^{-1}$ in the interval $[n^{-2}, 1-n^{-2}]$ $(n = 2, 3, \ldots)$. It is obvious that we can find a sequence $r_2, r_3, \ldots$ of positive integers satisfying the condition

(12) $$\omega_n\big(\omega_n'(r_n^{-1})\big) = o(n^{-1}) \quad \text{as} \quad n \to \infty.$$

Every such sequence associated with $\{F_t\}_{t>0}$ will be called *admissible*. It should be noted that for admissible sequences $r_2, r_3, \ldots$, by virtue of the inequality $\omega_n\big(\omega_n'(h)\big) \geqslant h$, the asymptotic relation

(13) $$r_n^{-1} = o(n^{-1}) \quad \text{as} \quad n \to \infty$$

holds.

As an example we shall present admissible sequences associated with semigroups of symmetric stable distributions. Consider a semi-group of distribution functions F_t with characteristic functions

$$\varphi_t(s) = \exp(-t|s|^a), \tag{14}$$

where a is a constant satisfying the inequality $0 < a \leqslant 2$. Of course, for $a = 2$ we have a semi-group of Gaussian distributions.

We shall prove that *each sequence* $r_2, r_3, \ldots$ *satisfying the condition*

$$\lim_{n\to\infty} r_n^{-1} n^{3+2/a} = 0 \quad \textit{if} \quad a < 2, \tag{15}$$

or the condition

$$\overline{\lim_{n\to\infty}}\, r_n^{-1} n^3 < \infty \quad \textit{if} \quad a = 2 \tag{16}$$

is admissible for a semi-group of symmetric stable laws with exponent a-

It is well-known ([4], p. 183) that each stable probability distribu. tion is absolutely continuous and its density function is bounded on the whole real line. Let $p(a, x)$ be the density function of $F_1(x)$. Since, by (14), $F_t(x) = F_1(xt^{-1/a})$, we have the inequality

$$\omega_n(h) = \omega_1(n^{1/a}h) \leqslant c_1 n^{1/a} h, \tag{17}$$

where c_1 is a constant. Furthermore, we have the equation for inverse functions $F_t^{-1}(x) = t^{1/a}F_1^{-1}(x)$. Hence we get the formula

$$\omega_n'(h) = n^{-1/a} \sup |F_1^{-1}(y_1) - F_1^{-1}(y_2)|,$$

where the supremum is extended over all y_1, y_2 satisfying the conditions $|y_1 - y_2| \leqslant h,\ n^{-2} \leqslant y_1,\ y_2 \leqslant 1-n^{-2}$. Since the distribution F_1 is symmetric and unimodal (see [7], [16]) the above supremum is not greater than $p(a, x_n)^{-1}h$, where x_n is defined by the equation

$$F_1(x_n) = 1 - n^{-2}. \tag{18}$$

Thus

$$\omega_n'(h) \leqslant n^{-1/a} p(a, x_n)^{-1} h. \tag{19}$$

For $a < 2$ there exists a positive constant c_2 such that

$$\lim_{x\to\infty} x^a\big(1 - F_1(x)\big) = c_2 \tag{20}$$

(see [9], p. 201 and [4], p. 182). Moreover, from a Wintner's result ([15]; see also [11]) we obtain an asymptotic formula

$$\lim_{x\to\infty} x^{1+a} p(a, x) = \frac{1}{\pi}\Gamma(1+a)\sin\frac{a\pi}{2}.$$

Hence and from (18) and (20) it follows that there exists a constant c_3 such that $p(\alpha, x_n)^{-1} \leqslant c_3 n^{2+2/\alpha}$ $(\alpha < 2)$. Thus, by (17) and (19),

$$\omega_n\big(\omega_n'(h)\big) \leqslant c n^{2+2/\alpha} h \quad (\alpha < 2),$$

where c is a constant. Hence it follows that a sequence $r_2, r_3, \ldots$ satisfying (15) is admissible for $\alpha < 2$.

If $\alpha = 2$, then

$$p(2, x) = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{x^2}{4}\right)$$

and $\lim_{x\to\infty} x\big(1-F_1(x)\big)\exp(x^2/4) = \pi^{-1/2}$ (see [3], p. 131). Hence and from (18) it follows that $p(2, x_n)^{-1} \leqslant c_3 x_n^{-1} n^2$, where c_3 is a constant. Thus, by (17) and (19),

$$\omega_n\big(\omega_n'(h)\big) \leqslant c x_n^{-1} n^2 h \quad (\alpha = 2),$$

where c is a constant. Since $\lim_{n\to\infty} x_n = \infty$, each sequence $r_2, r_3, \ldots$ satisfying (16) is, by the last inequality, admissible.

LEMMA 2. *Let $\{F_t\}_{t>0}$ be an admissible semi-group of continuous distribution functions and let $r_2, r_3, \ldots$ be an admissible sequence associated with this semi-group. If $s_1, s_2, \ldots$ is a sequence of integers satisfying the condition*

$$\lim_{n\to\infty} \frac{s_n}{n} = d > 0, \tag{21}$$

then for every real number x we have the formula

$$\lim_{n\to\infty} \sum F_{1/n}\left(x - \sum_{i=1}^{s_n} F_{1/n}^{-1}\left(\frac{k_i}{r_n+1}\right)\right)(r_n+1)^{-s_n} = F_d(x),$$

where the summation $\sum$ is extended over all systems $k_1, k_2, \ldots, k_{s_n}$ of integers satisfying the condition $1 \leqslant k_i \leqslant r_n$ $(i = 1, 2, \ldots, s_n)$.

Proof. For brevity we introduce the notation

$$A_n(x) = \sum F_{1/n}\left(x - \sum_{i=1}^{s_n} F_{1/n}^{-1}\left(\frac{k_i}{r_n+1}\right)\right)(r_n+1)^{-s_n}. \tag{22}$$

Let p_n and q_n be integers satisfying the conditions $p_n \geqslant 1$, $q_n \leqslant r_n$,

$$\frac{p_n-1}{r_n+1} < \frac{1}{n^2} \leqslant \frac{p_n}{r_n+1}, \quad \frac{q_n+1}{r_n+1} \leqslant 1 - \frac{1}{n^2} < \frac{q_n+2}{r_n+1}. \tag{23}$$

Put

$$(24)\qquad B_n(x) = \sum_{*} F_{1/n}\left(x - \sum_{i=1}^{s_n} F_{1/n}^{-1}\left(\frac{k_i}{r_n+1}\right)\right)(r_n+1)^{-s_n},$$

where the summation $\sum_{*}$ is extended over all systems $k_1, k_2, \ldots, k_{s_n}$ of integers satisfying the condition $p_n \leqslant k_i \leqslant q_n$ $(i = 1, 2, \ldots, s_n)$. By a simple reasoning we obtain the inequality

$$|A_n(x) - B_n(x)| \leqslant \sum_{j=1}^{s_n} \sum_{(j)} F_{1/n}\left(x - \sum_{i=1}^{s_n} F_{1/n}^{-1}\left(\frac{k_i}{r_n+1}\right)\right)(r_n+1)^{-s_n},$$

where the summation $\sum_{(j)}$ is running over all systems $k_1, k_2, \ldots, k_{s_n}$ of integers satisfying the conditions $1 \leqslant k_i \leqslant r_n$ $(i = 1, 2, \ldots, s_n)$ and $k_j \neq p_n, p_n+1, \ldots, q_n-1, q_n$. Hence we get the inequality

$$\begin{aligned} |A_n(x) - B_n(x)| &\leqslant s_n(p_n - 1 + r_n - q_n) r_n^{s_n-1}(r_n+1)^{-s_n} \\ &\leqslant s_n(p_n + r_n - q_n)(r_n+1)^{-1}. \end{aligned}$$

Finally, taking into account (13), (21) and (23), we obtain the formula

$$(25)\qquad \lim_{n\to\infty}\left(A_n(x) - B_n(x)\right) = 0.$$

Consider the expression

$$(26)\qquad C_n(x) = \sum_{*} \int_{a_{k_1}}^{a_{k_1+1}} \int_{a_{k_2}}^{a_{k_2+1}} \cdots \int_{a_{k_{s_n}}}^{a_{k_{s_n}+1}} F_{1/n}\left(x - \sum_{i=1}^{s_n} x_i\right) dF_{1/n}(x_1) dF_{1/n}(x_2) \ldots dF_{1/n}(x_{s_n}),$$

where

$$(27)\qquad a_k = F_{1/n}^{-1}\left(\frac{k}{r_n+1}\right).$$

Since

$$\int_{a_{k_i}}^{a_{k_i+1}} dF_{1/n}(x_i) = \frac{k_i+1}{r_n+1} - \frac{k_i}{r_n+1} = \frac{1}{r_n+1},$$

the expression (24) can be written in the form

$$B_n(x) = \sum_{*} \int_{a_{k_1}}^{a_{k_1+1}} \int_{a_{k_2}}^{a_{k_2+1}} \cdots \int_{a_{k_{s_n}}}^{a_{k_{s_n}+1}} F_{1/n}\left(x - \sum_{i=1}^{s_n} a_{k_i}\right) dF_{1/n}(x_1) dF_{1/n}(x_2) \ldots dF_{1/n}(x_{s_n}).$$

Thus

(28) $|B_n(x)-C_n(x)|$

$$\leqslant \sum_{*} \omega_n\Big(\sum_{i=1}^{s_n} |a_{k_{i+1}}-a_{k_i}|\Big) \int_{a_{k_1}}^{a_{k_1+1}} \int_{a_{k_2}}^{a_{k_2+1}} \cdots \int_{a_{k_{s_n}}}^{a_{k_{s_n}+1}} dF_{1/n}(x_1)\,dF_{1/n}(x_2)\ldots dF_{1/n}(x_{s_n}).$$

Since, by (23), the interval $[p_n/(r_n+1),\ q_n/(r_n+1)]$ is contained in the interval $[n^{-2}, 1-n^{-2}]$, we have the inequality

$$|a_{k_{i+1}}-a_{k_i}| = \left|F_{1/n}^{-1}\Big(\frac{k_i+1}{r_n+1}\Big)-F_{1/n}^{-1}\Big(\frac{k_i}{r_n+1}\Big)\right| \leqslant \omega_n'(r_n^{-1}),$$

whenever $p_n \leqslant k_i \leqslant q_n$. Thus, by (21) and by well-known formula $\omega_n'(mh) \leqslant m\omega_n'(h)$ $(m = 1, 2, \ldots)$, inequality (28) implies

$$|B_n(x)-C_n(x)| \leqslant s_n\,\omega_n\big(\omega_n'(r_n^{-1})\big) = dn\,\omega_n\big(\omega_n'(r_n^{-1})\big)+o(1),$$

which, by (12), yields

(29) $$\lim_{n\to\infty} \big(B_n(x)-C_n(x)\big) = 0.$$

Further, from (13), (21), (23), (26) and from the formula

$$F_{(s_n+1)/n}(x) = F_{1/n}^{*(s_n+1)}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} F_{1/n}\Big(x-\sum_{i=1}^{s_n} x_i\Big)dF_{1/n}(x_1)\,dF_{1/n}(x_2)\ldots dF_{1/n}(x_{s_n})$$

it follows that

$$0 \leqslant F_{(s_n+1)/n}(x)-C_n(x) \leqslant s_n\Big(\int_{-\infty}^{a_{p_n}} dF_{1/n}(y)+\int_{a_{q_n+1}}^{\infty} dF_{1/n}(y)\Big)$$
$$= s_n\Big(\frac{p_n}{r_n+1}+1-\frac{q_n+1}{r_n+1}\Big) = o(1).$$

Since, by (21), $\lim_{n\to\infty} F_{(s_n+1)/n}(x) = F_d(x)$, the last inequality together with (25) and (29) implies the assertion of the Lemma.

4. A combinatorial construction of relative processes. In this Chapter we shall give an effective combinatorial construction of relative processes with independent increments having continuous distribution functions.

THEOREM 3. *Let $\{F_t\}_{t>0}$ be an admissible semi-group of continuous distribution functions and let $r_2, r_3, \ldots$ be an admissible sequence of integers associated with this semi-group. For every $n \geqslant 2$ let $\langle k_{1j}^{(n)}, k_{2j}^{(n)}, \ldots, k_{r_nj}^{(n)}\rangle$,*

$j = 1, 2, \ldots, r_n^{r_n}$, *be a sequence of all ordered r_n-tuples of positive integers not exceeding r_n. Put* $a_n = r_n^{r_n}$, $b_n = \sum_{s=1}^{n} r_s^{1+r_s} r_{s+1}^{1+r_{s+1}}$ $(n \geqslant 2)$, $b_1 = 0$, $H(t) = 0$, *if* $t < 0$ *and* $H(t) = 1$ *if* $t \geqslant 0$. *Then the function*

$$f(t) = \sum_{n=2}^{\infty} \sum_{i=1}^{r_n} \sum_{j=1}^{a_n} \sum_{m=1}^{nr_{n+1}a_{n+1}} F_{1/n}^{-1}\left(\frac{k_{ij}^{(n)}}{r_n+1}\right) \times$$
$$\times H\left(t - b_{n-1} - \frac{(n-1)r_n a_n + (j-1)r_n + (i-1)}{n}\right)$$

is a relative process with independent increments. Moreover, $\{F_t\}_{t>0}$ *is the family of its distribution functions.*

Proof. Consider a system of intervals $I_p = [c_{p-1}, c_p)$ $(p = 1, 2, \ldots, k)$, where $c_0 = 0$. In what follows we assume that the index n satisfies the conditions $n \geqslant 2$ and $\min\limits_{1 \leqslant p \leqslant k} |I_p| > 2n^{-1}$. For every such index n we can define an auxiliary system of intervals

$$I_{pn} = \left[\frac{u_{pn}}{n}, \frac{v_{pn}}{n}\right) \quad (p = 1, 2, \ldots, k),$$

where u_{pn}, v_{pn} are integers,

$$u_{1n} = 0, \quad nc_{p-1} \leqslant u_{pn} \leqslant nc_{p-1}+1, \quad nc_p - 1 \leqslant v_{pn} \leqslant nc_p \quad (p = 1, 2, \ldots, k) \tag{30}$$

and

$$u_{p+1,n} = v_{pn}+1 \quad (p = 1, 2, \ldots, k-1). \tag{31}$$

Of course, $I_{pn} \subset I_p$ $(p = 1, 2, \ldots, k)$ and

$$\lim_{n\to\infty} |I_{pn}| = |I_p| \quad (p = 1, 2, \ldots, k). \tag{32}$$

Moreover, by (31), the distance between two consecutive intervals I_{pn} and $I_{p+1,n}$ is equal to n^{-1}.

Let us introduce the notation

$$U(n, m) = \left[b_{n-1} + \frac{(m-1)r_n a_n}{n}, b_{n-1} + \frac{mr_n a_n}{n}\right),$$

where $m = 1, 2, \ldots, nr_{n+1}a_{n+1}$ and $n = 2, 3, \ldots$. Further, for any system $y_1, y_2, \ldots, y_k$ of real numbers we put

$$A(n, m; y_1, y_2, \ldots, y_k) = \bigcap_{p=1}^{k} \{t\colon I_{pn}+t \subset U(n, m), f(I_{pn}+t) \leqslant y_p\}. \tag{33}$$

By the definition of the function f the distance between its consecutive jump points in the interval $U(n, m)$ is equal to n^{-1}. Put

$$w_{pn} = v_{pn} - u_{pn} \quad (p = 1, 2, \ldots, k). \tag{34}$$

If $I_{pn}+t_0$ is contained in $U(n,m)$, then the interval $I_{pn}+t_0$ contains exactly w_{pn} jump points of the function f. Furthermore, the same jump points belong to every interval $I_{pn}+t$, where t is taken from an interval of the lenght n^{-1} containing t_0. Thus as $n\to\infty$ we have

$$(35)\qquad |A(n,m;y_1,y_2,\dots,y_k)| = n^{-1}a(n,m;y_1,y_2,\dots,y_k)+O(n^{-1})$$

uniformly in m, where $a(n,m;y_1,y_2,\dots,y_k)$ is the number of all $\sum\limits_{p=1}^{k}w_{pn}$-tuples of consecutive jump points in the interval $U(n,m)$ such that the sum of w_{1n} first jumps is less or equal to y_1, the sum of next w_{2n} jumps is less or equal to y_2 and so on.

Now we shall establish an asymptotic formula for $a(n,m;y_1,y_2,\dots,y_k)$. The jump points of the function f in the interval $U(n,m)$ are of the form

$$b_{n-1}+\frac{(m-1)r_na_n+(j-1)r_n+(i-1)}{n}\qquad (i=1,2,\dots,r_n;\ j=1,2,\dots,a_n).$$

We note that the number of $\sum\limits_{p=1}^{k}w_{pn}$-tuples of consecutive jump points in $U(n,m)$ containing at least two jump points with different indices j is not greater than $a_n\sum\limits_{p=1}^{k}w_{pn}$, which is of order $o(r_na_n)$ uniformly in m as $n\to\infty$. Consequently, the number $a(n,m;y_1,y_2,\dots,y_k)$ is equal, with an accuracy $o(r_na_n)$, to the number of all $\sum\limits_{p=1}^{k}w_{pn}$-tuples of consecutive jump points in $U(n,m)$ corresponding to the same index j and satisfying the requirements formulated in the definition of $a(n,m;y_1,y_2,\dots,y_k)$. In other words, the number $a(n,m;y_1,y_2,\dots,y_k)$ is equal, with an accuracy $o(r_na_n)$, to the number of all pairs $\langle j,s\rangle$ $(j=1,2,\dots,a_n;\ s=0,1,\dots,r_n-\sum\limits_{p=1}^{k}w_{pn})$ for which the following inequalities are true:

$$\sum_{i=z_{p-1,n}+1}^{z_{pn}}F_{1/n}^{-1}\left(\frac{k_{i+s,j}^{(n)}}{r_n+1}\right)\leqslant y_p\qquad (p=1,2,\dots,k),$$

where

$$z_{0n}=0,\qquad z_{pn}=\sum_{q=1}^{p}w_{qn}\qquad (p=1,2,\dots,k).$$

Further, the last inequalities are equivalent to the following ones:

$$(36)\qquad k_{z_{p-1,n}+1+s,j}^{(n)}\leqslant (r_n+1)F_{1/n}\left(y_p-\sum_{i=z_{p-1,n}+2}^{z_{pn}}F_{1/n}^{-1}\left(\frac{k_{i+s,j}^{(n)}}{r_n+1}\right)\right)\qquad (p=1,2,\dots,k).$$

From the definition of r_n-tuples $\langle k_{1j}^{(n)}, k_{2j}^{(n)}, \ldots, k_{r_nj}^{(n)}\rangle$, by a combinatorial argument, it follows that for any fixed index s the number of indices j satisfying (36) is given by the formula

$$\sum r_n^{r_n-z_{kn}} \prod_{p=1}^{k} \left[(r_n+1) F_{1/n}\left(y_p - \sum_{i=z_{p-1,n}+2}^{z_{pn}} F_{1/n}^{-1}\left(\frac{d_{pi}}{r_n+1}\right)\right)\right],$$

where the summation is extended over all systems d_{pi} $(i = z_{p-1,n}+2, \ldots, z_{pn};\ p = 1, 2, \ldots, k)$ of integers satisfying the condition $1 \leqslant d_{pi} \leqslant r_n$ and $[x]$ denotes the integral part of x. Setting

$$(37) \qquad s_{pn} = z_{pn} - z_{p-1,n} - 1 = w_{pn} - 1 \quad (p = 1, 2, \ldots, k),$$

we can write the last expression in the form

$$a_n \prod_{p=1}^{k} \left[\sum_{(p)} F_{1/n}\left(y_p - \sum_{i=1}^{s_{pn}} F_{1/n}^{-1}\left(\frac{k_{pi}}{r_n+1}\right)\right) (r_n+1)^{-s_{pn}} \right] + o(a_n),$$

where the summation $\sum_{(p)}$ is extended over all systems $k_{p1}, k_{p2}, \ldots, k_{ps_{pn}}$ of integers satisfying the condition $1 \leqslant k_{pi} \leqslant r_n$ $(i = 1, 2, \ldots, s_{pn})$. Moreover, since this expression does not depend on the index s and $0 \leqslant s \leqslant \leqslant r_n - z_{kn}$, the number of pairs $\langle j, s\rangle$ satisfying (36) and, consequently, the number $a(n, m; y_1, y_2, \ldots, y_k)$ are given by the formula

$$a_n (r_n - z_{kn}) \prod_{p=1}^{k} \left[\sum_{(p)} F_{1/n}\left(y_p - \sum_{i=1}^{s_{pn}} F_{1/n}^{-1}\left(\frac{k_{pi}}{r_n+1}\right)\right) (r_n+1)^{-s_{pn}} \right] + o(r_n a_n).$$

From (32), (34) and (37) it follows that $\lim_{n\to\infty} s_{pn}/n = |I_p|$ and $z_{kn} = O(n)$, which, by (13), implies $z_{kn} = o(r_n)$. Thus, by Lemma 2 and formula (35),

$$(38) \qquad A(n, m; y_1, y_2, \ldots, y_k) = n^{-1} r_n a_n \prod_{p=1}^{k} F_{|I_p|}(y_p) + o(n^{-1} r_n a_n)$$

uniformly in m.

Now consider the set

$$(39) \qquad B_p(n, m; \varepsilon) = \{t;\ I_p + t \subset U(n\ m),\ |f(I_p+t) - f(I_{pn}+t)| > \varepsilon\},$$

where ε is a positive number. By (30) we conclude that if the interval $I_p + t$ is contained in $U(n, m)$, then the set $(I_p+t)\setminus(I_{pn}+t)$ contains at most two jump points of the function f. Thus the function f has a saltus of absolute magnitude greater than $\frac{1}{2}\varepsilon$ in the set $(I_p+t)\setminus(I_{pn}+t)$

whenever $t \in B_p(n, m; \varepsilon)$. Since, by (30)

$$|(I_p+t)\setminus(I_{pn}+t)| \leqslant 2n^{-1},$$

we have the inequality

$$|B_p(n, m; \varepsilon)| \leqslant 2n^{-1} b_p(n, m; \varepsilon), \tag{40}$$

where $b_p(n, m; \varepsilon)$ is the number of jumps of the function f in $U(n, m)$ of absolute magnitude greater than $\frac{1}{2}\varepsilon$. In other words, $b_p(n, m; \varepsilon)$ is equal to the number of integers $k_{ij}^{(n)}$ $(i = 1, 2, \ldots, r_n;\ j = 1, 2, \ldots, a_n)$ for which

$$\left| F_{1/n}^{-1}\left(\frac{k_{ij}^{(n)}}{r_n+1}\right)\right| > \tfrac{1}{2}\varepsilon.$$

Since the last inequality is equivalent to the union of two inequalities

$$k_{ij}^{(n)} < F_{1/n}(-\tfrac{1}{2}\varepsilon)(r_n+1), \qquad k_{ij}^{(n)} > F_{1/n}(\tfrac{1}{2}\varepsilon)(r_n+1),$$

we obtain by a simple combinatorial reasoning an estimation

$$b_p(n, m; \varepsilon) \leqslant a_n(r_n+1)\big(1-F_{1/n}(\tfrac{1}{2}\varepsilon)+F_{1/n}(-\tfrac{1}{2}\varepsilon)\big).$$

Hence, by (40), we get the inequality

$$|B_p(n, m; \varepsilon)| \leqslant 2n^{-1} a_n(r_n+1)\big(1-F_{1/n}(\tfrac{1}{2}\varepsilon)+F_{1/n}(-\tfrac{1}{2}\varepsilon)\big). \tag{41}$$

Further, setting

$$\begin{aligned} C_p(n, m) = \{t\colon I_{pn}+t \subset U(n, m),\ I_p+t \not\subset U(n, m)\}\cup \\ \cup\big(\{t\colon I_p+t \subset U(n, m)\}\setminus U(n, m)\big)\cup\big(U(n, m)\setminus \{t\colon I_p+t \subset U(n, m)\}\big), \end{aligned} \tag{42}$$

we have the inequality

$$|C_p(n, m)| \leqslant 2n^{-1}+2|I_p|. \tag{43}$$

For every positive number ε, taking into account (33), (39) and (42), we obtain the inclusions

$$U(n, m)\cap \bigcap_{p=1}^{k} \{t\colon f(I_p+t) < x_p\} \subset A(n, m; x_1+\varepsilon, x_2+\varepsilon, \ldots, x_k+\varepsilon)$$
$$\cup \bigcup_{p=1}^{k} B_p(n, m; \varepsilon)\cup \bigcup_{p=1}^{k} C_p(n, m),$$

$$A(n, m; x_1-\varepsilon, x_2-\varepsilon, \ldots, x_k-\varepsilon) \subset U(n, m)\cap \bigcap_{p=1}^{k} \{t\colon f(I_p+t) < x_p\}\cup$$
$$\cup \bigcup_{p=1}^{k} B_p(n, m; \varepsilon)\cup \bigcup_{p=1}^{k} C_p(n, m).$$

Hence and from (38), (41) and (43) we get the inequalities

$$(44)\qquad \left|U(n,m)\cap\bigcap_{p=1}^{k}\{t: f(I_p+t)<x_p\}\right|\leqslant n^{-1}r_n a_n\prod_{p=1}^{k}F_{|I_p|}(x_p+\varepsilon)+ \\ +2kn^{-1}r_n a_n\big(1-F_{1/n}(\tfrac{1}{2}\varepsilon)+F_{1/n}(-\tfrac{1}{2}\varepsilon)\big)+o(n^{-1}r_n a_n),$$

$$(45)\qquad \left|U(n,m)\cap\bigcap_{p=1}^{k}\{t: f(I_p+t)<x_p\}\right|\geqslant n^{-1}r_n a_n\prod_{p=1}^{k}F_{|I_p|}(x_p-\varepsilon)+ \\ +2kn^{-1}r_n a_n\big(F_{1/n}(\tfrac{1}{2}\varepsilon)-1-F_{1/n}(-\tfrac{1}{2}\varepsilon)\big)+o(n^{-1}r_n a_n)$$

uniformly in n.

By the definition of numbers a_n and b_n for every positive number T there exist integers N and M satisfying the conditions

$$b_{N-1}+\frac{Mr_N a_N}{N}\leqslant T<b_{N-1}+\frac{(M+1)r_N a_N}{N},\qquad 1\leqslant M\leqslant Nr_{N+1}a_{N+1}.$$

Since $b_{N-1}\geqslant r_N a_N r_{N-1}a_{N-1}$, we have $N^{-1}r_N a_N=o(b_{N-1})$ and consequently, $N^{-1}r_N a_N=o(T)$. Thus

$$T=b_{N-1}+\frac{Mr_N a_N}{N}+o(T).$$

Further, taking into account the decomposition

$$\left(0,b_{N-1}+\frac{Mr_N a_N}{N}\right)=\bigcup_{n=2}^{N-1}\bigcup_{m=1}^{nr_{n+1}a_{n+1}}U(n,m)\cup\bigcup_{m=1}^{M}U(N,m),$$

the formula $|U(n,m)|=n^{-1}r_n a_n$ and the limit relation for $\varepsilon>0$,

$$\lim_{n\to\infty}\big(1-F_{1/n}(\tfrac{1}{2}\varepsilon)+F_{1/n}(-\tfrac{1}{2}\varepsilon)\big)=0,$$

we obtain, by (44) and (45), the inequalities

$$\left|\bigcap_{p=1}^{k}\{t: f(I_p+t)<x_p\}\cap[0,T)\right|\leqslant T\prod_{p=1}^{k}F_{|I_p|}(x_p+\varepsilon)+o(T),$$

$$\left|\bigcap_{p=1}^{k}\{t: f(I_p+t)<x_p\}\cap[0,T)\right|\geqslant T\prod_{p=1}^{k}F_{|I_p|}(x_p-\varepsilon)+o(T).$$

Hence we get the formulas

$$\left|\bigcap_{p=1}^{k}\{t: f(I_p+t)<x_p\}\right|_{\overline{R}}\leqslant\prod_{p=1}^{k}F_{|I_p|}(x_p+\varepsilon),$$

$$\left|\bigcap_{p=1}^{k}\{t: f(I_p+t)<x_p\}\right|_{\underline{R}}\geqslant\prod_{p=1}^{k}E_{|I_p|}(x_p-\varepsilon),$$

which, by virtue of the arbitrariness of ε and the continuity of distribution functions $F_{|I_p|}$ $(p = 1, 2, \ldots, k)$, imply the equality

$$\left|\bigcap_{p=1}^{k} \{t: f(I_p+t) < x_p\}\right|_R = \prod_{p=1}^{k} F_{|I_p|}(x_p).$$

Thus the function f is a relative process with distribution functions $\{F_t\}_{t>0}$, which completes the proof.

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