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DIFFERENTIABILITY THEOREM FOR ELLIPTIC EQUATIONS CONSIDERED ON A COMPACT RIEMANNIAN MANIFOLD

BY

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The object of the present paper is to study the solutions of elliptic differential equations considered on a compact infinitely differentiable manifold M_n. It is well known that on such a manifold the Riemannian metric can be defined. Therefore we can assume without loss of generality that M_n is a Riemannian manifold and the differential operator is expressed in an invariant form in terms of the covariant derivatives. We use the method of Hilbert spaces with the so-called "negative norm" due to Lax [7] and utilized by him in the case of an elliptic operator with periodic coefficients considered on the class of periodic functions in Euclidean space. Such functions may be treated as defined on some special compact manifold, namely, on the Euclidean torus, so our result generalizes in some manner the theorem of Lax about periodic weak solutions of elliptic equations. The differentiability theorem we are going to prove follows from an a priori inequality, which is a consequence of the known inequality due to Gårding [4] (see section 3 of this paper). The essential point of the proof is based on the existence of smooth solution of the equation

$$(I+\Delta)^r\varphi = f$$

considered on the manifold M_n (1). This is assured by the known theorem of de Rham [10].

1. Preliminary remarks. Let us consider a compact infinitely differentiable manifold M_n with the Riemannian metric defined by a positive definite tensor field $g_{ik}(x)$. We suppose the coordinate system on M_n to be given by a finite covering with open sets,

$$M_n = \bigcup_{i=1}^{s} \Omega_i,$$

(1) Δ is the Laplace-Beltrami operator, I denotes the identical operator, f an infinitely differentiable function and r is a natural number.

each Ω_i being homeomorphic with some open region Ξ_i of the Euclidean space R^n. Let us denote by θ_i the corresponding homeomorphism: $\theta_i(\Omega_i) = \Xi_i$. Because the results which shall be proved are independent from the particular choice of the coordinate system, so it can be assumed without loss of generality that 1° each Ξ_i has a smooth boundary such that the estimate (3.2) is valid, and 2° $\inf\limits_{\substack{x\in\Omega_i \\ 1\leqslant i\leqslant s}} \det\|g_{jk}(\theta_i x)\|$ is positive.

The set of all infinitely differentiable complex tensor fields of order k over M_n shall be denoted by $C^{k,\infty}(M_n)$ (for $k=0$ we shall write simply $C^\infty(M_n)$ instead of $C^{0,\infty}(M_n)$). If f is an element of $C^{k,\infty}(M_n)$, then the tensor field of the covariant derivative of f of order l will be denoted by $\Delta^{(l)}f$ and its covariant components will be written as $\nabla_{\nu_1}\ldots\nabla_{\nu_l}f_{a_1\ldots a_k}$. For any two infinitely differentiable tensor fields f and h of order k their scalar product is defined by

$$(f,h)_{L_k^2(M_n)} \stackrel{\text{df}}{=} \int\limits_{M_n} f_{a_1\ldots a_k}\overline{h^{a_1\ldots a_k}}\,d_g x$$

($d_g x$ is the invariant measure on M_n given by the Riemannian metric). The completion of $C^{k,\infty}(M_n)$ in the norm $\|f\|_{L_k^2(M_n)} \stackrel{\text{df}}{=} (f,f)_{L_k^2(M_n)}^{1/2}$ is a Hilbert space. It will be denoted by $L_k^2(M_n)$ (²).

Because M_n is compact, the formula of "integration by parts" follows from the generalized theorem of Stokes. It will be used later in the form

$$(1.1)\qquad \int\limits_{M_n} (\nabla^{\nu_1}\ldots\nabla^{\nu_l}\Delta^{\mu_1}\ldots\nabla^{\mu_p}\varphi_{a_1\ldots a_k\nu_1\ldots\nu_l}) f^{a_1\ldots a_k}_{\mu_1\ldots\mu_p}\,d_g x$$
$$= (-1)^{l+p}\int\limits_{M_n}\varphi_{a_1\ldots a_k\nu_1\ldots\nu_l}\nabla^{\mu_p}\ldots\nabla^{\mu_1}\nabla^{\nu_l}\ldots\nabla^{\nu_1} f^{a_1\ldots a_k}_{\mu_1\ldots\mu_p}\,d_g x$$
$$(\varphi\in C^{k+l,\infty}(M_n),\ f\in C^{k+p,\infty}(M_n)).$$

For $\varphi\in C^{k,\infty}(M_n)$ and $s = 0,1,2,\ldots$, define the semi-norms by

$$|||\varphi|||_s \stackrel{\text{df}}{=} \sup_{x\in M_n}\left[\left(\nabla^{(s)}\varphi(x)\right)_{a_1\ldots a_{k+s}}\left(\nabla^{(s)}\overline{\varphi(x)}\right)^{a_1\ldots a_{k+s}}\right]^{1/2}.$$

In such a way the set of all infinitely differentiable tensor fields of order k becomes a locally convex space, which will be denoted by $\mathcal{D}_k(M_n)$. The linear functionals over $\mathcal{D}_k(M_n)$ will be called the *tensor-distributions* and the space of such functionals will be denoted by $\mathcal{D}'_k(M_n)$.

(²) It is readily to be seen that $L_k^2(M_n)$ is the direct integral of all tensor spaces of order k tangent to M_n.

It is easy to show that in the case of a compact manifold the semi-norms defined above in the set $C^\infty(M_n)$ yield the same topology as that defined in local terms by de Rham [10]. Thus $\mathscr{D}'_0(M_n)$ is the space of all distributions (in the sense of L. Schwartz) over M_n. The space $L^2_k(M_n)$ may be considered as a subspace of $\mathscr{D}'_k(M_n)$, because each $f \in L^2_k(M_n)$ defines the linear functional

$$\langle\varphi, f\rangle \overset{\mathrm{df}}{=} (\varphi, \bar{f})_{L^2_k(M_n)}, \quad \varphi \in \mathscr{D}_k(M_n),$$

over $\mathscr{D}_k(M_n)$.

For each $f \in \mathscr{D}'_k(M_n)$ and $l \leqslant k$, let us consider the tensor-distribution $F \in \mathscr{D}'_{k-l}(M_n)$ defined by

$$\langle\nabla^{(l)}\varphi, f\rangle \underset{\varphi \in D_{k-l}(M_n)}{\equiv} (-1)^l\langle\varphi, F\rangle. \tag{1.2}$$

We call F a *weak divergence of order* l of f and denote it by $\mathrm{div}^{(l)}_w f$.

In particular, if f is an infinitely differentiable tensor field of order k, then formula (1.1) yields

$$(\mathrm{div}^{(l)}_w f)_{\nu_{l+1}\ldots\nu_k} = \nabla^{\nu_1}\ldots\nabla^{\nu_l} f_{\nu_1\ldots\nu_l\nu_{l+1}\ldots\nu_k}.$$

For $l = k = 1$ this is the divergence (in the usual sense) of the vector field f.

We shall use later the estimate

$$\left|\int\limits_{M_n} a^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_l}\varphi_{\alpha_1\ldots\alpha_k}\psi^{\beta_1\ldots\beta_l} d_g x\right| \leqslant |||a|||_0 \|\varphi\|_{L^2_k(M_n)} \|\psi\|_{L^2_l(M_n)}, \tag{1.3}$$

valid for arbitrary $a \in C^{k+1,\infty}(M_n)$, $\varphi \in L^2_k(M_n)$, $\psi \in L^2_l(M_n)$. It may be proved in a simple manner by applying the Schwarz inequality to the integral on the left.

2. The spaces $H^p_k(M_n)$. In the present section we are going to define the Hilbert spaces which will be used later to study the solutions of differential equations. We assume that the index m will be always a natural number or zero and the index p will be an arbitrary integer.

Given two infinitely differentiable tensor fields f, h of order k, let us introduce their scalar product by

$$(f, h)_m \overset{\mathrm{df}}{=} \sum_{j=0}^{m} (\nabla^{(j)}f, \nabla^{(j)}h)_{L^2_{k+j}(M_n)}. \tag{2.1}$$

If we put

$$\|f\|_m \overset{\mathrm{df}}{=} (f, f)_m^{1/2},$$

the completion of the set $C^{k,\infty}(M_n)$ in the norm $\| \ \|_m$ becomes a Hilbert space, which we shall denote by $H_k^m(M_n)$ (evidently $H_k^0(M_n)$ is identical with $L_k^2(M_n)$). For $f \in C^{k,\infty}(M_n)$, we define also the norm with negativ index by

$$\|f\|_{-m} \overset{\text{df}}{=} \sup_{\varphi \in C^{k,\infty}(M_n)} \frac{\left|(\varphi, f)_{L_k^2(M_n)}\right|}{\|\varphi\|_m} \tag{2.2}$$

and consider the completion of $C^{k,\infty}(M_n)$ in the norm $\| \ \|_{-m}$, which we shall denote by $H_k^{-m}(M_n)$. (In the case $k = 0$ we shall write simply $H^p(M_n)$ instead of $H_0^p(M_n)$.) From (2.2) it follows

$$\left|(\varphi, f)_{L_k^2(M_n)}\right| \leqslant \|\varphi\|_m \|f\|_{-m}$$

for arbitrary $\varphi, f \in C^{k,\infty}(M_n)$. This implies that the scalar product in $L_k^2(M_n)$ can be extended to a bilinear form (which shall be denoted by $(\,,)$) defined on the Cartesian product $H_k^p(M_n) \times H_k^{-p}(M_n)$ and that the generalized Schwarz inequality

$$|(\varphi, f)| \leqslant \|\varphi\|_p \|f\|_{-p} \tag{2.3}$$

holds for all $\varphi \in H_k^p(M_n)$ and $f \in H_k^{-p}(M_n)$.

The set of all linear functionals over $H_k^m(M_n)$ of the form $(\varphi, f)_{L_k^2(M_n)}$ (where f is an arbitrary fixed tensor field of class $C^{k,\infty}(M_n)$) form a complete system of functionals [3]. Therefore the space $H_k^{-m}(M_n)$ may be identified with the space adjoint to $H_k^m(M_n)$ and therefore it is a Hilbert space. Because of the reflexivity also $H_k^m(M_n)$ may be treated as the space adjoint to $H_k^{-m}(M_n)$. The isomorphism between $H_k^p(M_n)$ and $\left(H_k^{-p}(M_n)\right)^*$ is given by the correspondence

$$H_k^p(M_n) \ni f \leftrightarrow l \in \left(H_k^{-p}(M_n)\right)^*$$

with

$$l(\varphi) = (\varphi, f)$$

identically for $\varphi \in H_k^{-p}(M_n)$.

For $f \in C^{k,\infty}(M_n)$ let

$$|f|_j \overset{\text{df}}{=} (\nabla^{(j)} f, \nabla^{(j)} f)^{1/2}_{L^2_{k+j}(M_n)}. \tag{2.4}$$

If a sequence $\{f_n\}$ of infinitely differentiable tensor fields of order k converges in the norm $\| \ \|_m$ to some $f \in H_k^m(M_n)$, then it is fundamental in the norm $\| \ \|_m$ and all the more in the norm $| \ |_j$ with $0 \leqslant j \leqslant m$. Because

(3) The set Λ of linear functionals on a Banach space X is said to be a *complete system of functionals* if $l(x) = 0$ for all $l \in \Lambda$ implies $x = 0$. If X is reflexive, then Λ is dense in the adjoint space X^*.

the space $L^2_{k+j}(M_n)$ is complete, there exists a tensor field (which shall be denoted by $\nabla^{(j)}_s f$) such that $\{f_n\}$ converges to $\nabla^{(j)}_s f$ in the norm $|\ |_j$. We call $\nabla^{(j)}_s f$ the *strong covariant derivative of order* j of the tensor field f. The formulas (2.1) and (2.4) are valid for all $f, h \in H^m_k(M_n)$, when the covariant derivation is understood in the strong sense.

PROPOSITION 2.1. *If $p_1 > p_2$, then the space $H^{p_1}_k(M_n)$ may be embedded into the space $H^{p_2}_k(M_n)$ algebraically and topologically.*

Proof. Consider at first p_1 and p_2 non-negative. Evidently $\|f\|_{p_1} \geqslant \|f\|_{p_2}$ for all $f \in C^{k,\infty}(M_n)$. To prove the compatibility (4) of the norms $\|\ \|_{p_1}$ and $\|\ \|_{p_2}$ on $C^{k,\infty}(M_n)$ it is sufficient to show that for each $p > 0$ and $f \in H^p_k(M_n)$ the condition $\Delta^{(0)}_s f = 0$ implies $\nabla^{(j)}_s f = 0$ for $j = 1, 2, \ldots, p$. Let φ be an infinitely differentiable tensor field of order $k+j$ over M_n and $\{f_n\}$ a sequence of infinitely differentiable tensor fields of order k tending to f in the norm $\|\ \|_p$. After integration by parts (formula (1.1)) we have

$$(\nabla^{(j)} f_n, \varphi)_{L^2_{k+j}(M_n)} = (-1)^j (f_n, \psi)_{L^2_k(M_n)}$$

with

$$\psi_{a_1 \ldots a_k} = \nabla^{\nu_j} \ldots \nabla^{\nu_1} \varphi_{\nu_1 \ldots \nu_j a_1 \ldots a_k}$$

and then in the limit

$$(2.5) \qquad (\nabla^{(j)}_s f, \varphi)_{L^2_{k+j}(M_n)} = (-1)^j (\nabla^{(0)}_s f, \psi)_{L^2_k(M_n)}.$$

Formula (2.5) proves our supposition, because $C^{k+j,\infty}(M_n)$ is dense in $L^2_{k+j}(M_n)$. Thus the space $H^{p_1}_k(M_n)$ can be treated as a (obviously dense) subset of the space $H^{p_2}_k(M_n)$, where $p_1 > p_2 \geqslant 0$. Accordingly the space $H^{-p_2}_k(M_n)$ can be also embedded into the space $H^{-p_1}_k(M_n)$, because they both may be considered as the spaces adjoint to $H^{p_2}_k(M_n)$ and $H^{p_1}_k(M_n)$ respectively. The proof is complete.

Let us study yet the relationship between the spaces $H^p_k(M_n)$ and the space of tensor-distributions $\mathscr{D}'_k(M_n)$ introduced in the preceding section. The convergence in $\mathscr{D}_k(M_n)$ is obviously stronger than the convergence in $H^p_k(M_n)$, thus each $f \in H^{-p}_k(M_n)$ defines a tensor distribution Jf according to the formula

$$\langle \varphi, Jf \rangle \overset{\mathrm{df}}{=} (\varphi, f).$$

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(4) We call two norms $\|\ \|_{(1)}$ and $\|\ \|_{(2)}$ *compatible* on the set Σ, if the following conditions hold:

1° $\|f\|_{(1)} \leqslant \|f\|_{(2)}$ for each $f \in \Sigma$,

2° every sequence $\{f_n\} \subset \Sigma$ fundamental in both norms $\|\ \|_{(1)}$ and $\|\ \|_{(2)}$ and tending to zero in the norm $\|\ \|_{(1)}$ tends to zero also in the norm $\|\ \|_{(2)}$.

It is known [5] that in such a case the completion of Σ in the norm $\|\ \|_{(2)}$ may be embedded in a one-to-one and continuous manner into the completion of Σ in the norm $\|\ \|_{(1)}$.

Clearly the mapping J is one-to-one. In particular, if $f \in L_k^2(M_n)$, then $Jf = \bar{f}$ almost everywhere on M_n.

PROPOSITION 2.2. *$f \in H_k^{-m}(M_n)$ if and only if the distribution Jf can be represented in the form*

$$Jf = \sum_{j=0}^{m} \operatorname{div}_w^{(j)} \psi_j \tag{2.4}$$

with $\psi_j \in L_{k+j}^2(M_n)$.

Proof. Let $f \in H_k^{-m}(M_n)$. From the duality of the spaces $H_k^m(M_n)$ and $H_k^{-m}(M_n)$ and from a theorem of F. Riesz applied to the space $H_k^m(M_n)$ it follows the existence of a $f_1 \in H_k^m(M_n)$ such that

$$(\varphi, f) = (\varphi, f_1)_m$$

identically for $\varphi \in C^{k,\infty}(M_n)$. According to the formula (2.1), this can be written in the form

$$(\varphi, f) = \sum_{j=0}^{m} (\nabla^{(j)} \varphi, \nabla_s^{(j)} f)_{L_{k+j}^2} \tag{2.5}$$

or

$$\langle \varphi, Jf \rangle = \sum_{j=0}^{m} < \nabla^{(j)} \varphi, \nabla_s^{(j)} \bar{f}_1 \rangle$$

and the latter identity yields (2.4) with $\psi_j = (-1)^j \nabla_s^{(j)} \bar{f}_1$. The converse statement is also easy to verify, if we observe that the expression on the right of (2.5) defines a linear functional over $C^{k,\infty}(M_n)$ which is continuous in the norm $\| \, \|_m$.

3. The differentiability theorem. In the present section we are going to study the existence and differentiability properties of the solutions of elliptic differential equations with infinitely differentiable coefficients considered on the manifold M_n. At first, we shall prove some differential inequalities, which are a simple consequence of the following well known theorem proved by Gårding [4]:

THEOREM. *Let Ω be a bounded domain of the Euclidean space R^n and let us consider the Dirichlet integral*

$$I(f) \overset{\text{df}}{=} \sum_{0 \leqslant |\alpha|, |\beta| \leqslant m} \int_{\Omega} a_{\alpha\beta}(x) D^{\alpha} f(x) D^{\beta} f(x) \, dx$$

with

$$D^{\alpha} f(x) \overset{\text{df}}{=} \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}, \quad \alpha = (\alpha_1, \ldots, \alpha_n), \ |\alpha| = \alpha_1 + \ldots + \alpha_n.$$

We suppose the coefficients $a_{\alpha\beta}$ to be infinitely differentiable complex-valued functions satisfying the conditions

1° $a_{\alpha\beta} = \overline{a_{\beta\alpha}}$ *for* $|\alpha| = |\beta| = m$,

2° $\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\eta^\alpha\eta^\beta \geqslant c(\eta_1^2+\ldots+\eta_n^2)^m$ *for arbitrary real* $\eta_1, \ldots, \eta_n$ *and arbitrary* $x\in\Omega$ *(with* c *independent of* x*).*

In such a case a positive constant t_0 *may be chosen so that for arbitrary* $f\in C_0^\infty(\Omega)$ (5) *and* $t\geqslant t_0$ *the inequality*

$$I(f)+t\|f\|^2_{L^2(\Omega)} \geqslant c\sum_{0\leqslant|\alpha|\leqslant m}\|D^\alpha f\|^2_{L^2(\Omega)} \tag{3.1}$$

holds with c *independent of* f (6).

We shall also use the estimate

$$\sum_{|\alpha|=1}\|D^\alpha f\|^2_{L^2(\Omega)} \leqslant ct^{-1-1/k}\left((t\|f\|^2_{L^2(\Omega)}+\sum_{|\alpha|=k}\|D^\alpha f\|^2_{L^2(\Omega)}\right) \quad (0\leqslant 1\leqslant k) \tag{3.2}$$

valid for Ω with sufficiently smooth boundary, if $f\in C^\infty(\Omega_0)$ (with $\Omega_0\supset\Omega$) and if t is sufficiently large, which is due to Ehrling [1].

PROPOSITION 3.1. *Consider the tensor field* $a\in C^{2m,\infty}(M_n)$ *satisfying the following conditions:*

$$a^{\alpha_1\ldots\alpha_m,\beta_1\ldots\beta_m} = \overline{a^{\beta_1\ldots\beta_m,\alpha_1\ldots\alpha_m}}, \tag{3.I}$$

(3.II) *for each* $x\in M_n$ *there is a constant* $c>0$ *such that*

$$\big(a(x)\big)^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_m}t^{\alpha_1}\ldots t^{\alpha_m}t_{\beta_1}\ldots t_{\beta_m} \geqslant c\left(\sum_{j=1}^n t_j^2\right)^m \tag{3.3}$$

for every real vector t *lying in the space tangent to* M_n *at the point* x*. (Because* M_n *is compact, one can obviously choose a constant* c *independent of* x *such that* (3.1) *holds). Under the above assumptions the inequality*

$$\int_{M_n} a^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_m}(\nabla_{\beta_2}\ldots\nabla_{\beta_m}f)(\nabla^{\alpha_1}\ldots\nabla^{\alpha_m}\bar f)\,d_gx + t\int_{M_n}|f|^2d_gx \tag{3.4}$$
$$\geqslant c\left(t\int_{\Xi_j}|f|^2d\xi+\sum_{|\alpha|=m}\int_{\Xi_j}|D^\alpha f|^2d\xi\right)\quad (\xi\overset{\mathrm{df}}{=}\theta_jx,\ x\in\Omega_j)$$

holds for $f\in C_0^\infty(\Omega_j)$ *with* $1\leqslant j\leqslant s$ *and* t *greater than some constant* t_0 *(independent of* f *and* j*).*

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(5) $C_0^\infty(\Omega)$ denotes the set of all infinitely differentiable functions with compact support contained in Ω.

(6) In this section, c denotes always a positive constant, may be taking different values in different inequalities.

To prove (3.4) it is sufficient to write the integral on the left in terms of the local coordinates and to apply inequality (3.1). Note that if, in particular, $a^{\beta_1\ldots\beta_m}_{a_1\ldots a_m} = \delta^{\beta_1}_{\beta_2}\ldots\delta^{\beta_m}_{a_m}$, then (3.4) yields

$$|f|^2_m + t|f|^2_0 \geqslant c\left(t\int\limits_{\Xi_j}|f|^2\,d\xi + \sum_{|a|=m}\int\limits_{\Xi_j}|D^a f|^2\,d\xi\right) \tag{3.5}$$

for $f\in C_0^\infty(\Omega_j)$, if $t\geqslant t_0$.

PROPOSITION 3.2. *For each* $f\in C^\infty(M_n)$, $0\leqslant l\leqslant m$ *and* $t\geqslant t_0$ *the inequality*

$$|f|^2_l \leqslant ct^{-1+l/m}(t|f|^2_0 + |f|^2_m) \tag{3.6}$$

holds (t_0 *is independent of* f).

Proof. For $f\in C_0^\infty(\Omega_j)$, $1\leqslant j\leqslant s$, it is easy to obtain (3.6) by writing $|f|^2_l$ in terms of local coordinates and making use of inequalities (3.2) and (3.5). Let now $f\in C^\infty(M_n)$ and let us consider a finite decomposition of unity $\{\varphi_j^2\}$ with the carrier of φ_j contained in Ω_j and $\sum\varphi_j^2 = 1$. At first, we shall prove by induction

$$|f|^2_l \leqslant ct^{-1+l/m}\left(t|f|^2_0 + |f|^2_m + \sum_j|\varphi_j f|^2_m\right), \tag{3.7}$$

$$f\in C^\infty(M_n), \quad 0\leqslant l\leqslant m.$$

For $l = 0$ the estimate (3.7) is evident. Suppose it is true for $l < r$ (with some $r\leqslant m$) and write $|f|^2_r$ in the form

$$\begin{aligned}|f|^2_r = \sum_j\Bigg\{&|\varphi_j f|^2_r + \sum_{l<r}\int\limits_{M_n}(b^j)^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_l}(\nabla_{\mu_1}\ldots\nabla_{\mu_r}\varphi_j f)(\nabla^{\nu_1}\ldots\nabla^{\nu_l}\bar f)\,d_g x + \\ &+ \sum_{l<r}\int\limits_{M_n}(c^j)^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_r}(\nabla_{\mu_1}\ldots\nabla_{\mu_l}f)(\nabla^{\nu_1}\ldots\nabla^{\nu_r}\varphi_j\bar f)\,d_g x + \\ &+ \sum_{k,l<r}\int\limits_{M_n}(d^j)^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l}(\nabla_{\mu_1}\ldots\nabla_{\mu_k}f)(\nabla^{\nu_2}\ldots\nabla^{\nu_l}\bar f)\,d_g x\Bigg\},\end{aligned}$$

where the tensor fields b^j, c^j, d^j are infinitely differentiable and have supports contained in Ω_j. Making use of the estimate (1.3), of inequality (3.6) applied to the function $\varphi_j f$ and of the inductive assumption we get (3.7) with $l = r$. It remains to estimate the sum $\sum\limits_j|\varphi_j f|^2_m$. Writing it in the form

$$|f|^2_m + \sum_j\sum_{\substack{k+l<2m\\0\leqslant k,l\leqslant m}}\int\limits_{M_n}(b^j)^{\nu_1\ldots\nu_k}_{\mu_1\ldots\mu_l}(\nabla_{\nu_1}\ldots\nabla_{\nu_k}f)(\nabla^{\mu_1}\ldots\nabla^{\mu_l}\bar f)\,d_g x$$

and applying (1.3) and (3.7) we obtain

$$\sum_j |\varphi_j f|_m^2 \leqslant |f|_m^2 + c\psi(t)\left(t|f|_0^2 + |f|_m^2 + \sum_j |\varphi_j f|_m^2\right)$$

with

$$\psi(t) = \sum_{k+j<2m} t^{-1+(k+j)/2m}.$$

Because $\psi(t) \to 0$ as $t \to \infty$, we get for sufficiently great t

$$\sum_j |\varphi_j f|_m^2 \leqslant c(t|f|_0^2 + |f|_m^2),$$

and this together with (3.7) yields (3.6).

PROPOSITION 3.3. *If $a \in C^{2m,\infty}(M_n)$ is the tensor field satisfying condition* (3.I) *and* (3.II), *then for every $f \in C^\infty(M_n)$ and $t \geqslant t_0$ the inequality*

$$(3.8) \qquad \int_{M_n} a_{\alpha_1\ldots\alpha_m}^{\beta_1\ldots\beta_m}(\nabla_{\beta_1}\ldots\nabla_{\beta_m}f)(\nabla^{\alpha_1}\ldots\nabla^{\alpha_m}\bar{f})\,d_gx + t|f|_0^2 \geqslant c(t|f|_0^2 + |f|_m^2)$$

holds (the constants t_0 and c are independent of f).

Proof. For $f \in C_0(\Omega_j)$ $(1 \leqslant j \leqslant s)$ the estimate (3.8) follows from proposition 3.1 if we make use of the inequality

$$t|f|_0^2 + |f|_m^2 \leqslant c\left(t\int_{\Xi_j} |f|^2 d\xi + \sum_{|\alpha|=m} \int_{\Xi_j} |D^\alpha f|^2 d\xi\right),$$

which is easy to be proved by the use of Schwarz inequality applied to the space $L^2(\Xi_j)$ and of the estimate (3.2). For an arbitrary $f \in C^\infty(M_n)$ one writes the left-hand side of (3.8) in the form

$$(3.9) \qquad \sum_j \int_{M_n} a_{\alpha_1\ldots\alpha_m}^{\beta_1\ldots\beta_m}\varphi_j(\nabla_{\beta_1}\ldots\nabla_{\beta_m}f)\varphi_j(\nabla^{\alpha_1}\ldots\nabla^{\alpha_m}\bar{f})\,d_gx + t\sum_j |\varphi_j f|_0^2,$$

where $\{\varphi_j^2\}$ is a finite decomposition of unity with the carrier of φ_j contained in Ω_j. Similar reasoning as that in the proof of the proposition 3.2 yields our assertion, if we make use of the estimate (3.6) and of the just proved inequality (3.8) with f replaced by $\varphi_j f$.

PROPOSITION 3.4. *Consider the differential operator of order $2m$*

$$Lf = \sum_{0\leqslant k,l\leqslant m} (-1)^k \nabla^{\alpha_1}\ldots\nabla^{\alpha_k}(a_{\alpha_1\ldots\alpha_k}^{\beta_1\ldots\beta_l}\nabla_{\beta_1}\ldots\nabla_{\beta_l}f) \qquad (f \in C^\infty(M_n))$$

defined on M_n with the coefficients infinitely differentiable and satisfying conditions (3.I) *and* (3.II) *(so L is elliptic and, moreover, uniformly elliptic, because M_n is compact). Let us put $D = I + \Delta$, where $\Delta f = -\nabla^\nu\nabla_\nu f$*

for $f \in C^\infty(M_n)$ and I is the identity operator. If r_0 is a positive integer, then for $0 \leqslant r \leqslant r_0$ and $f \in C^\infty(M_n)$ the inequalities

$$|(D^r Lf, f)| \geqslant c\|f\|^2_{m+r} \tag{3.10$_1$}$$

and

$$|(LD^r f, f)| \geqslant c\|f\|^2_{m+r} \tag{3.10$_2$}$$

hold, if the real part of the coefficient a in the term of L containing no derivatives has the lower bound greater than some positive number t_1 (depending on r_0 and not depending on f).

Proof. We shall prove inequality (3.10_1), the proof of (3.10_2) being analogous. Making use of the formulas for the covariant derivative of a product and of a commutant of two covariant derivatives one can bring the expression $D^r Lf$ to the form

$$\begin{aligned} D^r Lf = (-1)^{r+m} \nabla^{a_m} \ldots \nabla^{a_1} \nabla^{\nu_r} \ldots \nabla^{\nu_1} (a^{\beta_1 \ldots \beta_m}_{a_1 \ldots a_m} \nabla_{\nu_1} \ldots \nabla_{\nu_r} \nabla_{\beta_1} \ldots \nabla_{\beta_m} f) + \\ + f \operatorname{Re} a + \sum_{\substack{0 \leqslant k, l \leqslant m+r \\ 0 < k+l < 2(m+r)}} \nabla^{a_k} \ldots \nabla^{a_1} (b^{\beta_1 \ldots \beta_l}_{a_1 \ldots a_k} \nabla_{\beta_1} \ldots \nabla_{\beta_l} f) + \\ + \sum_{l \leqslant j \leqslant [r/2]} \Delta^j (\operatorname{Re} a \Delta^j f) + \sum_{0 \leqslant j \leqslant [(r-1)/2]} (-1) \Delta^j \nabla^\nu (\operatorname{Re} a \nabla_\nu \Delta^j f), \end{aligned} \tag{3.11}$$

where the tensor field $b^{\beta_1 \ldots \beta_l}_{a_1 \ldots a_k}$ does not depend on $\operatorname{Re} a$ (it depends only on the covariant derivatives of positive order of $\operatorname{Re} a$ and on the remaining coefficients of L as well as on their covariant derivatives). The Dirichlet integral corresponding to the two last terms on the right of (3.11) is non-negative and may be rejected when deriving desired estimate. After integrating by parts and making use of the propositions 3.2 and 3.3 one obtains

$$|(D^r Lf, f)| \geqslant \big(c - c_1 \psi(t_1)\big)(t_1 |f|^2_0 + |f|^2_{m+r})$$

with c_1 depending on the coefficients of L (with only exception of $\operatorname{Re} a$) and

$$\psi(t) = \sum_{0 < j < m+r} t^{-1+j/(m+r)}.$$

This yields (3.10_1) for t sufficiently large.

PROPOSITION 3.5. *Let L be an elliptic differential operator satisfying the assumptions of proposition 3.4. For an arbitrary integer p and $f \in C^\infty(M_n)$ the a priori inequality*

$$\|f\|_p \leqslant c\|Lf\|_{p-2m} \tag{3.12}$$

holds (with c independent of f).

We shall prove at first some lemmas.

LEMMA 3.1. *If $a \in C^{k,\infty}(M_n)$ and $f \in C^{\infty}(M_n)$, then for any integer p there is a positive constant c (independent of f) such that*

$$\|\psi\|_p \leqslant c\|f\|_{p+k} \tag{3.13}$$

with $\psi = a_{a_1\ldots a_k}\nabla^{a_1}\ldots\nabla^{a_k}f$.

Proof. For $p \geqslant 0$ our estimate follows simply from inequality (1.3) when making use of the formula for the covariant derivation of the product. Suppose now $p = -m < 0$ and let us consider at first the case $m \geqslant k$. Integrating by parts and using the generalized Schwarz inequality we get

$$\left|\int\limits_{M_n} \varphi(x)\overline{a_{a_1\ldots a_k}}\Delta^{a_1}\ldots\Delta^{a_k}\bar{f}\,d_g x\right| \leqslant \|h\|_{m-k}\|f\|_{-m+k}$$

with an arbitrary $\varphi \in C^{\infty}(M_n)$, where $h = \nabla^{a_k}\ldots\nabla^{a_1}(\bar{a}_{a_1\ldots a_k}\varphi)$ and the expression on the right is not greater than $c\|\varphi\|_m\|f\|_{-m+k}$, which follows from (3.13) by putting there $p = m-k \geqslant 0$. Let now $m < k$. Integration by parts yields the identity

$$\left|\int\limits_{M_n} \varphi(x)\overline{a_{a_1\ldots a_k}}\nabla^{a_1}\ldots\nabla^{a_k}\bar{f}\,d_g x\right| = \left|\int\limits_{M_n} (\nabla^{a_m}\ldots\nabla^{a_1}\varphi\overline{a_{a_1\ldots a_k}})(\nabla^{a_{m+1}}\ldots\nabla^{a_k}\bar{f})\,d_g x\right|.$$

If we take derivative of the product $\varphi\overline{a_{a_1\ldots a_k}}$, we can estimate the integral on the right by means of (1.3). Thus for each $p < 0$ we get

$$\left|\int\limits_{M_n} \varphi(x)\overline{a_{a_1\ldots a_k}}\nabla^{a_1}\ldots\nabla^{a_k}\bar{f}\,d_g x\right| \leqslant c\|\varphi\|_{-p}\|f\|_{p+k}$$

and this implies (3.13).

LEMMA 3.2. *For an arbitrary integer p and $f \in C^{\infty}(M_n)$ we have*

$$\|\nabla^{(1)}f\|_p \leqslant \|f\|_{p+1}. \tag{3.14}$$

Proof. For $p \geqslant 0$ inequality (3.14) is obvious. Suppose now $p = -m < 0$ and let $\varphi \in C^{1,\infty}(M_n)$. Integrating by parts we have

$$(\varphi, \nabla^{(1)}f)_{L_1^2(M_n)} = \int\limits_{M_n} (\nabla^{\nu}\varphi_{\nu})\bar{f}\,d_g x$$

and applying on the right the generalized Schwarz inequality one obtains

$$|(\varphi, \nabla^{(1)}f)_{L_1^2(M_n)}| \leqslant \|\psi\|_{m-1}\|f\|_{-m+1} \tag{3.15}$$

with $\psi = \nabla^{\nu}\varphi_{\nu}$. Note that

$$\|\psi\|_{m-1} \leqslant \|\varphi\|_m$$

for $m = 1, 2, \ldots$ because of the inequality

$$(\nabla^{a_1}\ldots\nabla^{a_k}\nabla^{\nu}\varphi_{\nu})(\nabla_{a_1}\ldots\nabla_{a_k}\nabla^{\nu}\bar{\varphi}_{\nu}) \leqslant (\nabla^{a_1}\ldots\nabla^{a_{k+1}}\varphi_{\nu})(\nabla_{a_1}\ldots\nabla_{a_{k+1}}\bar{\varphi}^{\nu}),$$

which is easy to prove in the orthonormal coordinate system. So inequality (3.15) yields

$$\left|(\varphi, \nabla^{(1)}f)_{L_1^2(M_n)}\right| \leqslant \|\varphi\|_m \|f\|_{-m+1}$$

and this implies (3.14).

LEMMA 3.3. *For arbitrary* $r = 1, 2, \ldots$ *and arbitrary function* $f \in C^\infty(M_n)$ *the equation*

$$D^r h = f \tag{3.16}$$

has a solution $h \in C^\infty(M_n)$.

Proof. At first, let us recall the concept of weak solution of a differential equation. We say that $u \in \bigcup_{q=-\infty}^{\infty} H^q(M_n)$ is a weak solution of the equation $Lu = v$ (L being some differential operator with infinitely differentiable coefficients and $v \in \bigcup_{q=-\infty}^{\infty} H^q(M_n)$) if the equation

$$(L^+\varphi, u) = (\varphi, v)$$

holds identically for $\varphi \in C^\infty(M_n)$ (L^+ denotes the differential operator formally adjoint to L). This is equivalent to the following statement: Ju is a solution of the equation $Lu = Jv$ with the operator L understood in the distributional sense (J denotes the embedding of $H^p(M_n)$ into $\mathscr{D}_0'(M_n)$ described in section 2). To prove our lemma let us consider at first the case $r = 1$. It will be shown that equation (3.16) has at least one weak solution; because of the known theorem of de Rham [10] it must be an infinitely differentiable function, if so is the right member. From the identity

$$(D\varphi, \varphi) = \|\varphi\|_1^2,$$

valid for arbitrary $\varphi \in C^\infty(M_n)$, we get, by applying the generalized Schwarz inequality, the following a priori inequality for the operator D:

$$\|\varphi\|_1 \leqslant \|D\varphi\|_{-1}. \tag{3.17}$$

Consider now the linear functional l defined on the set of all functions of the form $D\varphi$ with $\varphi \in C^\infty(M_n)$ as follows:

$$l(D\varphi) \overset{\text{df}}{=} (\varphi, f).$$

Inequality (3.17) implies that l is well defined and continuous in the norm $\| \ \|_{-1}$. Therefore it can be extended to a linear functional on the whole space $H^{-1}(M_n)$ according to the Banach-Hahn theorem. From the duality of the spaces $H^{-1}(M_n)$ and $H^1(M_n)$ it follows that there is an $h \in H^{-1}(M_n)$ such that

$$l(\psi) = (\psi, h)$$

for $\psi \in H^{-1}(M_n)$. Especially if $\psi = D\varphi$ with $\varphi \in C^\infty(M_n)$, this yields

$$(D\varphi, h) = (\varphi, f),$$

which means that h is the desired weak solution of (3.16). The case $r > 1$ follows by induction.

Let us prove now the proposition 3.5. In the case $p = m$ it follows simply from inequality (3.10_1) with $r = 0$ and from the generalized Schwarz inequality. Suppose now $p > m$, so $p = m + r$ with $r > 0$, and consider the expression $(D^r Lf, f)$ with $f \in C^\infty(M_n)$. Integrating by parts and applying lemmas 3.1 and 3.2 we obtain

$$|(D^r Lf, f)| = c\|Lf\|_{-m+r}\|f\|_{m+r}$$

and this, together with (3.10_1), yields our proposition. For $p = m - r$ (with $r > 0$) let us estimate the expression (Lf, h) with h given by lemma 3.3. The inequalities (3.10_2) and (2.3) imply

$$c\|h\|^2_{m+r} \leqslant |(Lf, h)| \leqslant \|Lf\|_{-m-r}\|h\|_{m+r} \tag{3.18}$$

and lemma 3.1 yields

$$\|f\|_{m-r} \leqslant \|h\|_{m+r}. \tag{3.19}$$

The inequalities (3.18) and (3.19) give (2.12) with $p = m - r$. The proof is complete.

PROPOSITION 3.6. *Let L be the differential operator considered in proposition 3.4 and suppose $p + m \leqslant p_0$. Then there exists a constant t_2 (depending on p_0) such that the set $\Gamma(L)$ of all functions Lf with $f \in C^\infty(M_n)$ is dense in $H^p(M_n)$, if the real part of the coefficient a is greater than t_2.*

LEMMA 3.4. *If A is some positive constant and t_2 is sufficiently large, then*

$$\left|\left((D^r + A)Lf, f\right)\right| \geqslant c\|f\|^2_{m+r}$$

for all $f \in C^\infty(M_n)$.

The proof is similar to the proof of proposition 3.4 and will be omitted.

To prove proposition 3.6 we begin with the case $p = -m$ and show that each linear functional on $H^{-m}(M_n)$ vanishing on Γ must vanish identically. If l is such a functional, it can be represented by

$$l(\varphi) = (\varphi, f_0)$$

with $f_0 \in H_m(M_n)$. Consider the bilinear form

$$b(\varphi, \psi) \stackrel{\text{df}}{=} (L\varphi, \psi)$$

defined for $\varphi \in C^\infty(M_n)$ and $\psi \in H^m(M_n)$. Because of the estimate

$$|b(\varphi, \psi)| \leqslant \|L\varphi\|_{-m} \|\psi\|_m \leqslant c\|\varphi\|_m \|\psi\|_m$$

it can be extended to a continuous bilinear form on the whole space $H^m(M_n)$. Moreover,

$$b(\varphi, f_0) = l(L\varphi)$$

for $\varphi \in C^\infty(M_n)$. Finally, b being continuous, it follows that

$$b(\varphi, f_0) = 0 \tag{3.20}$$

for all $\varphi \in H^m(M_n)$, especially for $\varphi = f_0$. Inequality (3.10) with $r = 0$ yields

$$|b(\varphi, \varphi)| \geqslant c\|\varphi\|_m^2$$

for all $\varphi \in H^m(M_n)$ and this together with (3.20) implies $\|f_0\|_m = 0$. Thus we have proved that $\Gamma(L)$ is dense in $H^{-m}(M_n)$. Therefore it is dense in each space $H^p(M_n)$ with $p < -m$. To examine the case $p = -m+r$ (with $0 < r \leqslant p_0$) note at first that by means of similar arguments as used above it can be shown that lemma 3.4 implies the density of the set $\Gamma\big((D^r+A)L\big)$ in the space $H^{-m-r}(M_n)$. Therefore to each $f \in C^\infty(M_n)$ and to every positive number ε an $f_\varepsilon \in C^\infty(M_n)$ can be chosen such that

$$\|(D^r+A)Lf_\varepsilon - (D^r+A)f\|_{-m-r} < \varepsilon. \tag{3.21}$$

If A is taken in such a manner that proposition 3.5 is true with L replaced by D^r+A, then (3.21) implies

$$\|Lf_\varepsilon - f\|_{-m+r} < \varepsilon$$

and the proof is complete.

Note that it follows from propositions 3.5, 3.6 and lemma 3.1 that an elliptic differential operator L with sufficiently large real part of the coefficient a considered on the set $C^\infty(M_n)$ defines a continuous mapping from the space $H^p(M_n)$ onto some dense subset of $H^{p-2m}(M_n)$ and the inverse mapping is also continuous. So we have proved

THEOREM 1. *Let L be an elliptic operator of order $2m$ with infinitely differentiable coefficients satisfying conditions* (3.I) *and* (3.II) *and let us suppose that* $\inf_{x \in M_n} \operatorname{Re} a(x)$ *is greater than the numbers t_1, t_2 considered in propositions* 3.4 *and* 3.6. *Then the closure $\tilde{L}$ of the operator L (considered as the mapping from $H^{p+2m}(M_n)$ into $H^p(M_n)$) is a topological isomorphism of the space $H^{p+2m}(M_n)$ onto the space $H^p(M_n)$.*

Theorem 1 implies that to each $f \in H^p(M_n)$ there is some $u \in H^{p+2m}(M_n)$ such that $\tilde{L}u = f$. We call u the *strong solution* of the equation

$$Lu = f, \quad f \in H^p(M_n). \tag{3.22}$$

Obviously, each strong solution of (3.22) is always a weak one and the converse statement is also true, if L satisfies the assumptions of theorem 1. Consider now an arbitrary elliptic operator L (without assuming that $\inf_{x\in M_n} \operatorname{Re} a(x)$ is large). Then theorem 1 is valid for the operator $L_t \overset{\text{df}}{=} L+tI$ (with sufficiently large real constant t) and thus it can be deduced that if u is a weak solution of (3.22) with L replaced by L_t, then it must lie in $H^{p+2m}(M_n)$. The last statement yields the following differentiability theorem for the operator L:

THEOREM 2. *If L is an elliptic operator of order $2m$ with infinitely differentiable coefficients satisfying conditions* (3.I) *and* (3.II) *and if u is a weak solution of* (3.22), *then u must lie in $H^{p+2m}(M_n)$.*

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