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ON COMPACTIFICATION OF ABSOLUTE RETRACTS

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1. Problem and result. The notions of absolute retract (AR) and absolute neighbourhood retract (ANR) due to Borsuk [1] were specialized by Hanner [4] (see also Dowker [2]) for classes of spaces as follows:

Given a class τ of spaces, a space $X \in \tau$ is called an *absolute retract* for τ (AR_τ) — respectively *absolute neighbourhood retract* for τ (ANR_τ) — if, whenever X is imbedded as a closed subset of a space $Z \in \tau$, X is a retract of Z — respectively X is a retract of some neighbourhood U of X in Z.

Pełczyński [8] posed a problem, which may be precised as follows:

Let α and β be two classes of spaces and X a metric space. Is it true that if X is an AR_α, then there exists a compactification X^* of X such that X^* is an AR_β?

The purpose of this note is to give a solution of Pełczyński's problem in the case that compactification is a Hausdorff space, α denotes an arbitrary class of spaces which contains the class of all separable metric spaces and is contained in the class of all normal spaces, and β denotes any class of spaces having below formulated property (4). In particular, the class of all separable metric spaces as well the class of all metric spaces and the class of all normal spaces, has this property.

The answer is negative. Namely, we shall prove the following

THEOREM. *Let L_n be the rectiligne segment of ends $a = (0, 1)$ and $p_n = (1/n, 0)$. The space*

$$N = \bigcup_{n=1}^{\infty} L_n \tag{1}$$

has two following properties:

(2) *N is absolute retract for every class of spaces which contains the class of all separable metric spaces and is contained in the class of all normal spaces;*

(3) *every Hausdorff space Y containing N as a dense subset is not locally connected in any point $p \in Y-N$.*

Before we prove the Theorem, we shall deduce from it two corollaries which are true for every class τ of spaces having the following property:

(4) *for every space $X \in \tau$ there exists a locally connected space $Z \in \tau$ such that X is a closed subset of Z.*

Since open subset of a locally connected space is locally connected ([5], p. 106) and retraction preserves local connectedness ([1], p. 155), then, by (3), we have

COROLLARY 1. *If N is a dense subset of a Hausdorff space Y and if Y is an absolute neighbourhood retract for some class τ of spaces with property* (4), *then* $Y = N$.

Consequently, N being not compact, we have

COROLLARY 2. *If a Hausdorff space N^* is a compactification of N then N^* is not an absolute retract for any class τ of spaces with property* (4)

The announced solution is given by thesis (2) of Theorem and by Corollary 2.

2. Lemmas. If a class σ of spaces is contained in a class τ, $X \in \sigma$ and $X \in \mathrm{AR}_\tau$ (respectively $X \in \mathrm{ANR}_\tau$), then obviously $X \in \mathrm{AR}_\sigma$ (respectively $X \in \mathrm{ANR}_\sigma$).

Hanner has shown [4] that a normal space X is an absolute retract (respectively an absolute neighbourhood retract) for normal spaces if and only if for any normal space Y, any closed subset B of Y and any mapping $f\colon B \to X$ there exists an extension $f^*\colon Y \to X$ of f (respectively there exists an extension $\tilde{f}\colon U \to X$ of f over some neighbourhood U of B).

These remarks imply that for every class of spaces which contains the class of all separable metric spaces and is contained in the class of all normal spaces the definitions of absolute retract and of absolute neighbourhood retract, quoted at the beginning of the paper, are equivalent to definitions expressed in terms of extension of functions.

Using the latter definitions of absolute retract and absolute neighbourhood retract it is easy to show

LEMMA 1. *If A is a retract of X and X is absolute retract* (*respectively absolute neighbourhood retract*) *for some class of spaces which contains the class of all separable metric spaces and is contained in the class of all normal spaces, then A is an absolute retract* (*respectively an absolute neighbourhood retract*) *for the same class of spaces.*

The next two lemmas are concerned with Hausdorff spaces.

Lemma 2. *If A is a compact subset of X and if X is a dense subset of a Hausdorff space Y, then $A = \bar{A}$ and $A \cap \overline{X-A} = A \cap \overline{Y-A}$ (the closure is in Y, of course).*

Proof. It is well known (see for instance [5], p. 38) that in a Hausdorff space compact sets are closed. Hence $A = \bar{A}$.

The hypothesis $X \subset Y$ implies that

$$A \cap \overline{X-A} \subset A \cap \overline{Y-A}. \tag{5}$$

Conversely, if $p \in A \cap \overline{Y-A}$, then we have $U \cap (Y-A) \neq 0$ for every open neighbourhood U of p in Y. Since the set $U \cap (Y-A)$ is open in Y and, by hypothesis, the set X is dense in Y, we have $[U \cap (Y-A)] \cap X \neq 0$. Therefore $U \cap (X-A) \neq 0$ and this means that $p \in \overline{X-A}$. Hence

$$A \cap \overline{X-A} \supset A \cap \overline{Y-A}. \tag{6}$$

Both these inclusions, (5) and (6), imply the second equality of our Lemma.

Denoting now by $\mathrm{Fr}_Z(B)$ the boundary of B in the space Z, we easily obtain from Lemma 2

Lemma 3. *If A is a compact subset of N and if N is a dense subset of a Hausdorff space Y, then $\mathrm{Fr}_N(A) = \mathrm{Fr}_Y(Z)$.*

3. Proof of the Theorem. Since N is a separable metric space it suffices, in order to prove (2), to show that

(7) N is an absolute retract for normal spaces.

We shall do exactly this. Firstly notice that

(8) N is an absolute retract for separable metric spaces.

In fact, consider N as a subset of the triangle T of vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$, and remove from T all points $(0, y)$, where $0 \leqslant y < 1$. Denoting the resulting set by W and the interior (in the plane) of the triangle T by $\mathring{T}$, we have $\mathring{T} \subset W \subset T$. This implies, by a theorem of Fox ([3], Corollary), that W is an absolute retract for separable metric spaces. We now construct the obvious retraction of W onto N, and then, in view of Lemma 1, we obtain (8).

But N is clearly a G_σ set in the plane, i.e. an absolute G_δ set. Hence and from (8) we infer by using Hanner's theorem ([4], Theorem 4.2) that (7) is true.

It remains to prove (3). Let Y be a Hausdorff space containing N as a dense subset, $p \in Y-N$ and let U be an arbitrary open neighbourhood in Y of p such that

$$a \notin U. \tag{9}$$

We shall show that U is not connected, i.e. that Y is not locally connected at the point p.

Since $U \cap L_n \subset L_n$ and L_n is compact, we have by Lemma 2, $\overline{U \cap L_n} \subset L_n$. It follows by (1) and the hypothesis $p \in Y - N$ that $p \notin \overline{U \cap L_n}$ for any $n = 1, 2, \ldots$ and therefore

$$U \cap L_n \neq 0 \quad \text{for infinitely many } n. \tag{10}$$

Now let n_0 be a natural number such that $U \cap L_{n_0} \neq 0$ and let q be an arbitrary point of $U \cap L_{n_0}$. Denote by P the component of q in the set $U \cap N$ and by A its closure in N. Then we have

$$A \cap U = P, \tag{11}$$

$$A = \overline{P} \cap N. \tag{12}$$

In view of (9) we have also $P \subset N - (a)$, whence, P being a component in N, we have $P \subset L_{n_0}$. It follows by (12) that A is an arc contained in L_{n_0}. It is clear that

$$U - P \neq 0, \tag{13}$$

$$\mathrm{Fr}_N(A) \subset A - U, \tag{14}$$

$$P = A - \mathrm{Fr}_N(A). \tag{15}$$

By virtue of the formula $\mathrm{Int}_Y(A) = A - \mathrm{Fr}_Y(A)$ ([6], p. 29, formula (3)) and by Lemma 3 we infer from (15) that P is an open subset of Y, which implies that

$$P \text{ is an open subset of } U, \tag{16}$$

because $P \subset U$ and U is an open subset of Y by hypothesis.

Now let Q be the component of q in U. We shall prove that

$$P = Q. \tag{17}$$

Indeed, since P is the component of q in $U \cap X$, $P \subset Q$. On the other hand, in view of (14) and Lemma 3 we have $Q \cap \mathrm{Fr}_Y(A) = 0$, which implies that $Q \subset A$, because $q \in A \cap Q$ and Q is connected ([7], p. 80). Multiplying the inclusion $Q \subset A$ by U we obtain in view of (11) that $Q \subset P$.

We have thus proved (17).

By virtue of (16) and (17) the set Q is open in U. As a component of U, Q is also closed in U ([7], p. 81). Hence, in view of $q \in Q$, Q is non-void, and it is closed-open in U. But in view of (13) and (17) the complementary set $U - Q$ is at the same time non-void and closed-open in U. This means that U is not connected.

REFERENCES

[1] K. Borsuk, *Sur les retractes*, Fundamenta Mathematicae 17 (1931), p. 152-170.

[2] C. H. Dowker, *On a theorem of Hanner*, Arkiv för Matematik 2 (1952), p. 307-313.

[3] R. H. Fox, *A characterization of absolute neighbourhood retracts*, Bulletin of the American Mathematical Society 48 (1942), p. 271-275.

[4] O. Hanner, *Solid spaces and absolute retracts*, Arkiv för Matematik 1 (1952), p. 375-382.

[5] J. G. Hocking and G. S. Young, *Topology*, Reading and London 1961.

[6] C. Kuratowski, *Topologie I*, 3rd edition, Warszawa 1952.

[7] — *Topologie II*, 2nd edition, Warszawa 1952.

[8] A. Pełczyński, *Problem 594*, New Scottish Book, Wrocław 26. V. 1962 (non published).

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