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ALGEBRAS UNDER A MINIMAL NORM

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An algebra A over the real field R is a vector space over R closed with respect to a product xy which is linear in both x and y and such that the condition $\lambda(xy) = (\lambda x)y = x(\lambda y)$ holds for any $\lambda \in R$ and $x, y \in A$. The product is not necessarily associative. We assume that the algebra A contains a unit element e, i. e., an element satisfying the equality $ex = xe = x$ for any $x \in A$. Given any subset B of A, by $\dim B$ we denote the linear dimension of B, i. e., the power of a maximal set of linearly independent elements of B. By $[B]$ we denote the linear set spanned by the elements of B. For arbitrary elements $x_1, x_2, \ldots, x_n$ by $A(x_1, x_2, \ldots, x_n)$ we shall denote the subalgebra generated by $x_1, x_2, \ldots, x_n$. An algebra is called *normed* if it is a normed space over R under a *submultiplicative norm* $\| \ \|$, i. e., a norm satisfying in addition to the usual requirements the condition $\|xy\| \leqslant \|x\| \cdot \|y\|$ for any x and y in A. A norm $\| \ \|$ is called *multiplicative* if $\|xy\| = \|x\| \cdot \|y\|$ for every $x, y \in A$. A submultiplicative norm $\| \ \|$ is called *minimal* if $\|x^2\| = \|x\|^2$ for any $x \in A$.

The subalgebra A_r of an algebra A is called a *two-sided ideal* if $x_r x$, $xx_r \in A_r$ for every $x_r \in A_r$ and every $x \in A$. An algebra A is defined to be the *direct sum* of the two-sided ideals A_r $(r = 1, 2, \ldots,)$ if every element $x \in A$ can be uniquely represented in the form $x = \sum_r x_r$, where $x_r \in A_r$ $(r = 1, 2, \ldots)$ and the sum $\sum_r x_r$ contains a finite number of non-zero components. If A is the direct sum of two-sided ideals A_r $(r = 1, 2, \ldots)$, we write $A = \sum_r A_r$. For an x in A we write $x = \sum_r x_r$ with $x_r \in A_r$.

Let $A = \sum_r A_r$; then for every pair of indices s, t $(s \neq t)$ and for any $x_s \in A_s$, $x_t \in A_t$ the equation

$$x_s x_t = x_t x_s = 0 \tag{1}$$

holds. Indeed, by the definition of two-sided ideal, we have $x_s x_t$, $x_t x_s \in A_r$ $(r = s, t)$. Consequently, the supposition $x_s x_t \neq 0$ or $x_t x_s \neq 0$ contradicts the uniqueness of the representation of $x_s x_t$ or $x_t x_s$ as the sum of elements of the subalgebras A_r $(r = 1, 2, \ldots)$. As a direct consequence of (1) we obtain

$$xy = \Big(\sum_r x_r\Big)\cdot\Big(\sum_r y_r\Big) = \sum_r x_r y_r \tag{2}$$

and

$$x^2 = \sum_r x_r^2. \tag{3}$$

We note that if A is an algebra with the unit element e, then the direct sum $A = \sum_r A_r$ contains only a finite number of ideals A_r. Indeed, let $e = \sum_r e_r$ with $e_r \in A_r$, and suppose that for an index s we have $e_s = 0$. Then, by (2), for any non-zero $x_s \in A_s$

$$x_s = x_s e = x_s \sum_r e_r = x_s e_s = 0,$$

which is impossible. Consequently, $e_r \neq 0$ for all r. Since, by the definition of the direct sum, the sum $e = \sum_r e_r$ contains only a finite number of non-zero elements, there is only a finite number of A_r's in the sum $\sum_r A_r = A$.

THEOREM 1. *Let A be a normed algebra with a minimal norm. If A is the direct sum of ideals A_r $(r = 1, 2, \ldots, n)$, then for every $x \in A$ the equation*

$$\|x\| = \Big\|\sum_{r=1}^{n} x_r\Big\| = \max_{1\leqslant r\leqslant n}\|x_r\|. \tag{4}$$

holds.

PROOF. We prove the theorem for $n = 2$. The general case is easily deduced from this by induction. Suppose $A = A_1 + A_2$. Then any x in A can be written in the form $x = x_1 + x_2$, where $x_1 \in A_1$, $x_2 \in A_2$. If $x = 0$, then $x_1 = x_2 = 0$, so the theorem is true for $x = 0$. Let $x \neq 0$ and let $\|x_1\| \geqslant \|x_2\|$. Since, by (1), $x_1 x_2 = 0$, and since $x_2 = x - x_1$, we have $x_1(x - x_1) = 0$, whence

$$x_1 x = x_1^2. \tag{5}$$

The norm is minimal, so from (5) we infer

$$\|x_1\|^2 = \|x_1^2\| = \|x_1 x\| \leqslant \|x_1\|\cdot\|x\|. \tag{6}$$

Since $x_1 \neq 0$ (because otherwise $\|x_2\| \leqslant \|x_1\| = 0$ and, consequently, $x = 0$), from (6) we obtain the inequality $\|x_1\| \leqslant \|x\|$.

Define $(x)^{2^1} = x^2$, $(x)^{2^{k+1}} = \big((x)^{2^k}\big)^2$. According to (3), for every integer m we have

$$(x)^{2^m} = (x_1 + x_2)^{2^m} = (x_1)^{2^m} + (x_2)^{2^m}.$$

Hence for every integer m

$$\|(x)^{2^m}\| \leqslant \|(x_1)^{2^m}\| + \|(x_2)^{2^m}\|. \tag{7}$$

Since $\|x\| \neq 0$ and the norm is minimal, (7) gives

$$\|x\|^{2^m} \leqslant \|x_1\|^{2^m} + \|x_2\|^{2^m},$$

and

$$1 \leqslant \left(\frac{\|x_1\|}{\|x\|}\right)^{2^m} + \left(\frac{\|x_2\|}{\|x\|}\right)^{2^m} \tag{8}$$

for every integer m. Suppose $\|x_1\| < \|x\|$. Then, by (8),

$$1 \leqslant \lim_{m\to\infty}\left[\left(\frac{\|x_1\|}{\|x\|}\right)^{2^m} + \left(\frac{\|x_2\|}{\|x\|}\right)^{2^m}\right] = 0$$

which is a contradiction. Consequently,

$$\|x\| = \|x_1\| = \max(\|x_1\|\,\|x_2\|),$$

as required.

Henceforth by K we shall denote the unit ball of a normed space, i. e., the set $\{x: \|x\| \leqslant 1\}$ and by S the unit sphere, i. e., the boundary of K. The unit ball K of a normed space is said to be *rotund* ([2], p. 111) if every open segment in K is disjoint from its boundary S. In particular, the unit ball of a space whose dimension is equal to one is rotund.

LEMMA 1. *If the unit ball K of a normed linear space L is rotund, then at least one of the inequalities*

$$\|x - y\| > \|x\|, \qquad \|x + y\| > \|x\|$$

is valid for any pair of elements $x, y \in L$ $(y \neq 0)$.

Proof. Of course our Lemma is true for $x = 0$. Let us suppose that there exists a pair of non-zero elements $x, y \in L$ such that

$$\|x - y\| \leqslant \|x\|, \tag{9}$$

$$\|x + y\| \leqslant \|x\|. \tag{10}$$

Let

$$u = \frac{1}{\|x\|}(x - y), \qquad v = \frac{1}{\|x\|}(x + y).$$

Since $y \neq 0$, $u \neq v$. By (9) and (10), $u, v \in K$. Therefore for any $a \in R$, $0 < a < 1$, $[au + (1-a)v] \in K$, i. e. the open interval (u, v) is contained in K. Since $x/\|x\| = \frac{1}{2}(u+v)$ and $u \neq v$, we have $x/\|x\| \in (u, v)$. But $x/\|x\| \in S$, and this contradicts the definition of a rotund ball. Lemma 1 is thus proved.

THEOREM 2. *Let A be a normed algebra under a minimal norm. Suppose that A, treated as a linear space, is the direct sum of linear subspaces $A_1, A_2, \ldots, A_n$, and*

$$\Big\| \sum_{r=1}^{n} x_r \Big\| = \max_{1 \leq r \leq n} \|x_r\| \qquad (x_r \in A_r). \tag{11}$$

Furthermore, suppose that for every index r the unit ball $K \cap A_r$ of the subspace A_r is rotund. Then all the subspaces A_r are two-sided ideals of A and the algebra A is the direct sum of subalgebras $A_1, A_2, \ldots, A_n$.

The proof of Theorem 2 is based on the following lemmas. Henceforth we denote by x_r or $(x)_r$ the component of x belonging to the subspace A_r.

LEMMA 2. *For any $x \in S$ there exists an index r such that*

$$\|(x^2)\|_r = 1.$$

Lemma 2 is a direct consequence of the fact that the norm is minimal and of formula (11).

LEMMA 3. *Let a_r denote an arbitrary element for which*

$$\|a_r\| = 1 \tag{12}$$

and let for an index s

$$\|(a_r^2)_s\| = 1. \tag{13}$$

Then for any $t \neq r$ and for every x_t

$$(x_t^2)_s = 0. \tag{14}$$

Proof. Without loss of generality we may suppose that

$$\|x_t\| = 1. \tag{15}$$

Let us suppose that there exists an index $t \neq r$ and an element $x_t \in S \cap A_t$, for which $(x_t^2)_s \neq 0$. Since the ball $K \cap A_s$ is rotund, we may apply Lemma 1 to the elements $(a_r^2)_s \mp (x_t^2)_s$. In accordance with (13), this shows that at least one of the above elements has a norm greater than 1. Let e. g. $\|(a_r^2)_s - (x_t^2)_s\| > 1$. A similar argument shows that at least one of the inequalities

$$\|(a_r^2)_s - (x_t^2)_s \mp (a_r x_t - x_t a_r)_s\| \geq \|(a_r^2)_s - (x_t^2)_s\| > 1$$

is valid. Accordingly, let

$$\|[(a_r + x_t) \cdot (a_r - x_t)]_s\| = \|(a_r^2)_s - (x_t^2)_s - (a_r x_t - x_t a_r)_s\| > 1.$$

But this is impossible. In fact, since $t \neq r$, by (11), (12) and (15) we have

$$\|[(a_r + x_t) \cdot (a_r - x_t)]_s\| \leq \|(a_r + x_t) \cdot (a_r - x_t)\|$$
$$\leq \|a_r + x_t\| \cdot \|a_r - x_t\| \leq (\max \|a_r\|, \|x_t\|)^2 = 1.$$

If we argue similarly, we see that the assumption $(x_t^2)_s \neq 0$ leads to a contradiction in all the other possible cases.

LEMMA 4. *Let a_r denote an arbitrary element belonging to $S \cap A_r$ and let s denote such an index that $(a_r^2)_s = 1$. Then for any $t \neq r$ and every $x_t \in A_t$*

$$(a_r x_t)_s = (x_t a_r)_s = 0.$$

The proof of Lemma 4 is analogous to that of Lemma 3. We consider the elements of the form

$$(a_r^2)_s \mp (x_t a_r \mp a_r x_t)_s, \tag{16}$$

where $x_t \in S \cap A_t$ $(t \neq r)$ and we show that, for the norm of one of the elements (16), one of the assumptions $(a_r x_t)_s \neq 0$ or $(x_t a_r)_s \neq 0$ implies inequalities which are impossible.

LEMMA 5. *For every s_0 there exists one, and only one, r_0 such that for any $x_{r_0} \in S \cap A_{r_0}$ the equation $\|(x_{r_0}^2)_{s_0}\| = 1$ holds.*

Proof. Let us consider an arbitrary element $a = \sum_{r=1}^{n} a_r$, where for each r $\|a_r\| = 1$. By Lemma 2 for each index r there exists a non-empty set $E(a_r)$ of integers such that for any $s \in E(a_r)$ we have $\|(a_r^2)_s\| = 1$. From Lemma 3 it follows that for any pair r_1, r_2 $(r_1 \neq r_2)$ the relation $E(a_{r_1}) \cap E(a_{r_2}) = 0$ holds. Indeed, if $\|(a_{r_1}^2)_{s_1}\| = 1$, then $\|(a_{r_2}^2)_{s_1}\| = 0$. By $\{1, 2, \ldots, n\}$ we denote the set of integers $1, 2, \ldots, n$. Since $\bigcup_{r=1}^{n} E(a_r) \subset \{1, 2, \ldots, n\}$ and $\bigcup_{r=1}^{n} E(a_r)$ has n non-empty disjoint components, it follows that each of the sets $E(a_r)$ contains one and only one of the numbers $1, 2, \ldots, n$, and each of them belongs to one of the sets $E(a_r)$. Hence, for each s_0 there exists an integer r_0 such that $\|(a_{r_0}^2)_{s_0}\| = 1$. In virtue of Lemma 3, $E(x_{r_0}) \neq E(a_r)$ for any $r \neq r_0$ and every $x_{r_0} \in S \cap A_{r_0}$. Therefore $E(x_{r_0}) = E(a_{r_0})$, and Lemma 5 is thus proved.

LEMMA 6. *If $r \neq s$, then for any $x_r \in A_r$, $x_s \in A_s$*

$$x_r x_s = x_s x_r = 0.$$

Proof. Without loss of generality we can suppose that $\|x_r\| = \|x_s\| = 1$. Clearly, it is sufficient to show that the equation

$$(17)\qquad (x_r x_s)_t = (x_s x_r)_t = 0$$

is valid for any index t. By Lemma 5, for every t there exists an index m such that the equation

$$(18)\qquad \|(x_m^2)_t\| = 1$$

holds for every $x_m \in S \cap A_m$. If $m = r$ or $m = s$, then (17) follows from Lemma 4. Now suppose that $m \neq r$ and $m \neq s$. As in the proof of Lemma 3 we show the truth of the inequalities

$$(19)\qquad \|[(x_m + x_r \mp x_s)\cdot(x_m + x_r \mp x_s)]_t\| \leqslant 1.$$

According to (18), by Lemmas 3 and 4 we have

$$(20)\qquad [(x_m + x_r \mp x_s)\cdot(x_m + x_r \mp x_s)]_t = (x_m^2)_t \mp (x_r x_s)_t \mp (x_s x_r)_t.$$

If anyone of equations (17) were not true, then, as in the proof of Lemma 3, we could show that at least one of the elements (20) has a norm greater than one, but this contradicts inequality (19). Lemma 6 is thus proved.

LEMMA 7. *For every index r and for any $x_r \in A_r$, x_r^2 belongs to A_r.*

Proof. First we show that $e_s \neq 0$ for any s, e_s being the s-th component of the unit element e belonging to the subspace A_s. Indeed, by Lemma 6,

$$(21)\qquad x_s = x_s e = \sum_{r=1}^{n} x_s e_r = x_s e_s.$$

Therefore $e_s \neq 0$. Substituting in (21) $x_s = e_s$ we obtain the equation

$$(22)\qquad e_s = e_s^2$$

valid for any s.

Since the norm is minimal, by (22), we have

$$(23)\qquad \|e_s\| = \|e_s^2\| = \|e_s\|^2$$

for every index s. But $e_s \neq 0$, so, by (23), $\|e_s\| = 1$ holds for any index s. This means that $e_r \in S \cap A_r$ for each index r. Moreover, for any r we obtain the equation

$$(24)\qquad \|(e_r^2)_r\| = \|(e_r)_r\| = \|e_r\| = 1.$$

Let x_{r_0} denote a fixed element of a fixed subspace A_{r_0}. According to (24), by Lemma 3, we obtain the equation $\|(x_{r_0}^2)_s\| = 0$ valid for each $s \neq r_0$. Hence

$$x_{r_0}^2 = \sum_{s=1}^{n} (x_{r_0}^2)_s = (x_{r_0}^2)_{r_0} \in A_{r_0}.$$

Lemma 7 is thus proved.

LEMMA 8. *For any $x_r, y_r \in A_r$ $(r = 1, 2, \ldots, n)$ we have $x_r y_r$, $y_r x_r \in A_r$.*

Proof. By Lemma 7, there is nothing to prove whenever $y_r = \gamma x$ $(\gamma \in R, r = 1, 2, \ldots, n)$. Suppose that there exist indices r, s $(r \neq s)$, and linearly independent elements a_r and b_r such that $(a_r b_r)_s \neq 0$. Since $(a_r b_r)_s \neq 0$, at least one of the elements $(a_r b_r)_s \mp (b_r a_r)_s$ is different from zero. Let, then,

$$(25)\qquad (a_r b_r - b_r a_r)_s \neq 0.$$

We assume that

$$(26)\qquad \|a_r\| \leqslant \tfrac{1}{2}, \qquad \|b_r\| \leqslant \tfrac{1}{2}.$$

(If this were not true, we should consider the elements $a_r' = a_r/2\|a_r\|$, $b_r' = b_r/2\|b_r\|$ in place of a_r and b_r, respectively). Formula (26) implies the inequality

$$(27)\qquad \|a_r \mp b_r\| \leqslant \|a_r\| + \|b_r\| = 1.$$

Therefore, by (11), (24) and (27),

$$(28)\qquad \begin{aligned} &\|\{[e_s + (a_r - b_r)]\cdot[e_s \mp (a_r + b_r)]\}_s\| \\ &\leqslant \|[e_s + (a_r - b_r)]\cdot[e_s \mp (a_r + b_r)]\| \\ &\leqslant \|e_s + (a_r - b_r)\|\cdot\|e_s \mp (a_r + b_r)\| \\ &\leqslant \max(\|e_s\|, \|a_r - b_r\|)\cdot\max(\|e_s\|, \|a_r + b_r\|) = 1. \end{aligned}$$

As $r \neq s$, by Lemmas 6 and 7, the elements $(a_r^2)_s$, $(b_r^2)_s$, $e_s\cdot(a_r + b_r)$, $(a_r - b_r)\cdot e_s$ are equal to zero. Hence, by (22), we obtain

$$(29)\qquad \{[e_s + (a_r - b_r)]\cdot[e_s \mp (a_r + b_r)]\}_s = e_s \mp (a_r b_r - b_r a_r)_s.$$

We recall that the ball $K \cap A_s$ is rotund. Hence, according to (25) and (24), at least one of the elements (29) has a norm greater than one, but this contradicts (28).

In the same way we show that the suppositions $(a_r b_r)_s \neq 0$ or $(b_r a_r)_s \neq 0$ $(r \neq s)$ in all the remaining cases yield contradictions. Lemma 8 is thus proved.

Proof of Theorem 2. We have shown that all $x_r^2, y_r^2, x_r y_r, y_r x_r$ belong to A_r for any $x_r, y_r \in A_r$ $(r = 1, 2, \ldots, n)$. Hence the subspace A_r forms a subalgebra. By Lemma 6, A_r is a two-sided ideal for every index r. In other words, the algebra A is the direct sum of the two-sided ideals A_r $(r = 1, 2, \ldots, n)$.

Now we consider two examples which show that some of the assumptions of Theorem 2 are essential.

Example 1. Let A be an algebra generated by elements $a_1, a_2, \ldots, a_n$ $(n \geqslant 2)$. Let $a_{n+1} = a_1$. The multiplication is defined by the formulas

$$(30)\qquad \begin{aligned} a_r^2 &= a_{r+1} \quad (r = 1, 2, \ldots, n), \\ a_r a_s &= a_s a_r = 0 \quad \text{if} \quad r \neq s, \end{aligned}$$

and the norm by the formula

$$(31)\qquad \Big\| \sum_{r=1}^{n} a_r a_r \Big\| = \max_{1 \leqslant r \leqslant n} |a_r| \qquad (a_r \in R).$$

A straightforward verification shows that the algebra A does not contain the unit element. In fact, by (30), we have

$$a_1 \sum_{r=1}^{n} a_r a_r = a_1 a_2 \neq a_1$$

for any element $\sum_{r=1}^{n} a_r a_r$ in A.

On the other hand, we show that A satisfies all the remaining assumptions of Theorem 2. Clearly, $\| \; \|$ satisfies all the conditions postulated in the definition of a norm. Moreover,

$$\Big\| \sum_{r=1}^{n} a_r a_r \cdot \sum_{r=1}^{n} \beta_r a_r \Big\| = \Big\| \sum_{r=1}^{n} a_r \beta_r a_{r+1} \Big\| = \max_{1 \leqslant r \leqslant n} |a_r \beta_r|$$

$$\leqslant \max_{1 \leqslant r \leqslant n} |a_r| \cdot \max_{1 \leqslant r \leqslant n} |\beta_r| = \Big\| \sum_{r=1}^{n} a_r a_r \Big\| \cdot \Big\| \sum_{r=1}^{n} \beta_r a_r \Big\|.$$

Consequently, $\| \; \|$ is a submultiplicative norm. It is minimal too, because

$$\Big\| \Big(\sum_{r=1}^{n} a_r a_r \Big)^2 \Big\| = \Big\| \sum_{r=1}^{n} a_r^2 a_{r+1} \Big\| = \max_{1 \leqslant r \leqslant n} |a_r^2| = [\max_{1 \leqslant r \leqslant n} |a_r|]^2 = \Big\| \sum_{r=1}^{n} a_r a_r \Big\|^2.$$

Let $L_r = [a_r]$ denote the one-dimensional subspace spanned by the element a_r. Of course A is the direct sum of the subspaces L_r, and the unit balls $K \cap L_r$ are rotund $(r = 1, 2, \ldots, n)$. Formula (11) is also satisfied. For, by (31), for each index r we have $\|a_r\| = 1$. Hence

$$\Big\| \sum_{r=1}^{n} a_r a_r \Big\| = \max_{1 \leqslant r \leqslant n} |a_r| = \max_{1 \leqslant r \leqslant n} \|a_r a_r\|.$$

But the subspaces L_r are not subalgebras, because by formulas (30) $a_r^2 = a_{r+1} \notin L_r$. The existence of the unit element is thus an essential condition in Theorem 2.

Now we show that the assumption that the norm $\| \; \|$ is minimal cannot be omitted.

Example 2. We define in the complex field C the norm $\| \; \|$ by the formula

$$(32)\qquad \|ae + \beta i\| = |a| + |\beta|$$

(e denotes the unit element, i — the imaginary unit). Clearly enough, $\| \; \|$ satisfies all the conditions of the definition of a norm. Moreover, $\|e\| = 1$ and the fact that $\| \; \|$ is a submultiplicative norm, is easily verified as follows:

$$\begin{aligned} \|(a_1 e + \beta_1 i)\cdot(a_2 e + \beta_2 i)\| &= \|(a_1 a_2 - \beta_1 \beta_2) e + (a_1 \beta_2 + a_2 \beta_1) i\| \\ &= |a_1 a_2 - \beta_1 \beta_2| + |a_1 \beta_2 + a_2 \beta_1| \leqslant |a_1 a_2| + |\beta_1 \beta_2| + |a_1 \beta_2| + |a_2 \beta_1| \\ &= (|a_1| + |\beta_1|)(|a_2| + |\beta_2|) = \|a_1 e + \beta_1 i\| \cdot \|a_2 e + \beta_2 i\|. \end{aligned}$$

We consider one-dimensional subspaces $L_1 = [e_1]$, $L_2 = [e_2]$ generated by the elements $e_1 = \frac{1}{2}(e - i)$, $e_2 = \frac{1}{2}(e + i)$, respectively. Since the elements e_1, e_2 are linearly independent, the algebra C is the direct sum of the subspaces L_1 and L_2. In virtue of the fact that $\dim L_1 = \dim L_2 = 1$, we see that the unit balls $K \cap L_1$ and $K \cap L_2$ are rotund. Formula (11) is satisfied too; in fact,

$$\begin{aligned} \|\xi_1 e_1 + \xi_2 e_2\| &= \tfrac{1}{2}\|(\xi_1 + \xi_2) e + (\xi_2 - \xi_1) i\| \\ &= \tfrac{1}{2}(|\xi_1 + \xi_2| + |\xi_2 - \xi_1|) = \max(|\xi_1|, |\xi_2|). \end{aligned}$$

However, the norm $\| \; \|$ is not minimal, because, for example,

$$\|e + i\| = 2, \qquad \|(e + i)^2\| = \|2i\| = 2 \neq \|e + i\|^2.$$

Since L_1 and L_2 are not subalgebras of the field C, Theorem 2 is not true in our case. Hence the condition that the norm is minimal is essential in Theorem 2.

The following two theorems concern the theory of associative algebras. We now list some well-known facts of the theory of associative algebras. An element $a \in A$ is called *nilpotent* if $\lim_{n \to \infty} \sqrt[n]{\|a^n\|} = 0$. An element $a \in A$

is said to be *properly nilpotent* if both xa and ax are nilpotent for every x of A. The set consisting of all properly nilpotent elements of an algebra A is called a *radical*. An algebra is called *semi-simple* if its radical contains only the zero element.

The ideals different from the zero-ideal and from the whole algebra are called *proper ideals*. An algebra is said to be *simple* if it does not contain a proper ideal. Every finite-dimensional associative semi-simple algebra is the direct sum of two-sided simple ideals ([1], p. 38).

The algebra A is said to be a *division algebra* if for every a, b in A, with $a \neq 0$, the equations $ax = b$ and $ya = b$ are solvable in A. Let $B(b_1, b_2, \ldots, b_m)$ and $C(c_1, c_2, \ldots, c_t)$ be m and t dimensional subalgebras of an algebra A such that every element of B commutes with every one of C. Moreover, let the mt products $b_r c_s$ form a basis of A. Then we shall call algebra A the *direct product* of subalgebras B and C and we shall write $A = B \times C$. Every simple algebra A is expressible as $M \times D$, where M is a total matrix algebra and D is a division algebra ([1], p. 39, Theorem 9).

LEMMA 9. *A normed associative algebra A whose norm $\| \;\|$ is minimal contains no nilpotent elements different from zero.*

Proof. By the definition of the minimal norm, the equation

$$\sqrt[2^n]{\|(x)^{2^n}\|} = \sqrt[2^n]{\|x\|^{2^n}} = \|x\|$$

is valid for any $x \in A$ and every integer n. Consequently, for every $x \neq 0$, $\lim_{n\to\infty} \sqrt[n]{\|x^n\|} \neq 0$.

COROLLARY. *A normed associative algebra A whose norm is minimal is a semi-simple algebra.*

Two norms $\| \;\|_1$ and $\| \;\|_2$ defined in A are *equivalent* if there exist two positive numbers m and M such that $m\|x\|_1 \leqslant \|x\|_2 \leqslant M\|x\|_1$ for any $x \in A$. It is well known that any two norms in a finite dimensional linear space are equivalent.

LEMMA 10. *If $\| \;\|_1$ and $\| \;\|_2$ are two minimal norms in a finite dimensional algebra A, then for every x in A the equation $\|x\|_1 = \|x\|_2$ holds.*

Proof. Suppose on the contrary that there exists an element a in A such that $\|a\|_1 < \|a\|_2$. Of course $a \neq 0$. Let us consider the element $b = a/\|a\|_2$. We have

$$\|b\|_1 = \frac{\|a\|_1}{\|a\|_2} = q < 1, \qquad \|b\|_2 = 1.$$

Since the norms $\| \;\|_1$ and $\| \;\|_2$ are minimal, we have

$$\|(b)^{2^n}\|_1 = \|b\|_1^{2^n} = q^{2^n}, \qquad \|(b)^{2^n}\|_2 = \|b\|_2^{2^n} = 1$$

for any integer n. Hence

$$\lim_{n\to\infty} \|(b)^{2^n}\|_1 = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(b)^{2^n}\|_2 = 1.$$

But this is impossible, because the norms $\| \;\|_1$ and $\| \;\|_2$ are equivalent. Lemma 10 is thus proved.

Since in each of the algebras: the real field R, the complex field C, and the quaternion algebra Q, the multiplicative norm is minimal, Lemma 10 implies

COROLLARY. *Any minimal norm in each of the algebras: the real field R, the complex field C, and the quaternion algebra Q, is multiplicative.*

THEOREM 3. *If A is an associative, finite-dimensional normed algebra whose norm is minimal, then*

1) *A is the direct sum of subalgebras A_r $(r = 1, 2, \ldots, n)$ each of which is isometrically isomorphic to one of the following algebras: the real field, the complex field, the quaternion algebra.*

2) *For any $x = \sum_{r=1}^{n} x_r$ $(x_r \in A_r)$ we have*

$$\|x\| = \max_{1\leqslant r\leqslant n} \|x_r\|.$$

Proof. By Corollary of Lemma 9 the algebra A is semi-simple. Moreover, it is associative and finite-dimensional. Consequently, it is the direct sum of two-sided simple ideals A_r $(r = 1, 2, \ldots, n)$. Every subalgebra A_r is the direct product of a division algebra D_r and a total matrix algebra M_r ([1], p. 39). Since A_r $(r = 1, 2, \ldots, n)$ does not contain any non-zero nilpotents, the total matrix algebra M_r is isomorphic to the real field (the argument is that any matrix the elements of which are all zero except for the one that lies outside the principal diagonal, is a nilpotent element in the total matrix algebra). Therefore, every subalgebra A_r is a division algebra. By the well-known Frobenius Theorem ([3], X, § 52) every A_r is isomorphic to one of the following: the real field, the complex field, the quaternion algebra. The existence of the desired isometrical isomorphism follows from Lemma 10. The second part of our theorem is a consequence of Theorem 1.

THEOREM 4. *Let A be an associative finite-dimensional normed algebra under a minimal norm. If the group of isometries of A preserving the unit sphere is finite, then the algebra A is the direct sum of real fields.*

Proof. By Theorem 3, $A = \sum_{r=1}^{n} A_r$ where each A_r $(r=1,2,\ldots,n)$ is isometrically isomorphic to one of the following algebras: the real field R, the complex field C, the quaternion algebra Q. Suppose to the contrary that there exists a subalgebra A_s, A_1 say, which is not isomorphic to the real field. We examine the transformation T_u of algebra A defined by the formula

$$T_u\Big(\sum_{r=1}^{n} x_r\Big) = ux_1 + \sum_{r=2}^{n} x_r,$$

where u denotes any element belonging to $S \cap A_1$. The transformations T_u are isometries preserving the unit sphere S. In fact, in virtue of Corollary of Lemma 10 the minimal norm in each of the algebras R, C, Q, is multiplicative; so, since $u \in S$, we have $\|ux_1\| = \|u\| \cdot \|x_1\| = \|x_1\|$. Hence, by Theorem 3, we obtain the equation

$$\Big\| T_u\Big(\sum_{r=1}^{n} x_r\Big)\Big\| = \Big\| ux_1 + \sum_{r=2}^{n} x_r \Big\| = \max_{1 \leqslant r \leqslant n} \|x_r\| = \Big\|\sum_{r=1}^{n} x_r\Big\|.$$

Since A_1 is a division algebra, different transformations T_u correspond to different elements $u \in S \cap A_1$. Since $\dim A_1 \geqslant 2$, there exist infinitely many elements $u \in S \cap A_1$. Accordingly, there exist infinitely many isometries that transform the unit sphere S onto itself, contrary to the assumption. Theorem 4 is thus proved.

REFERENCES

[1] A. A. Albert, *Structure of algebras*, New York 1939.
[2] M. M. Day, *Normed linear spaces*, Berlin 1958.
[3] А. Д. Курош, *Курс высшей алгебры*, Москва 1949.

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A PROOF OF THE WELL-ORDERING THEOREM

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The usual proofs of the well-ordering theorem proceed by induction. It is also well known how to avoid induction and ordinal numbers in the proof. However, the resulting arguments are rather lengthy. Here we present a proof of this kind which we believe is still very short.

Let S be a non-void set and let $\mathscr{P}$ stand for "power-set of." By the axiom of choice there exists a mapping $\gamma: (\mathscr{P}S - \{\emptyset\}) \to S$ such that $\gamma Z \in Z$ for every $Z \in \mathscr{P}S - \{\emptyset\}$. Let Z^+ denote $Z - \{\gamma Z\}$.

We define a mapping $f: \mathscr{P}^2 S \to \mathscr{P}^2 S$ by

(1) $$f\mathscr{Z} = \{\bigcap_{Z \in \mathscr{U}} Z \,|\, \emptyset \neq \mathscr{U} \subset \mathscr{Z}\} \cup \{Z^+ \,|\, \emptyset \neq Z \in \mathscr{Z}\},$$

i. e. for $\mathscr{Z} \in \mathscr{P}^2 S$, $f\mathscr{Z}$ consists of all intersections of non-void sets of elements of $\mathscr{Z}$ (considered as subsets of S) as well as of all subsets of S obtained from elements Z ($Z \neq \emptyset$) of $\mathscr{Z}$ by removing from them their element γZ.

Next, we define a mapping $\varphi: \mathscr{P}S \to \mathscr{P}^2 S$ by

(2) $$\varphi Z = \bigcap_{\mathscr{Z} \supset \{Z\} \cup f\mathscr{Z}} \mathscr{Z}.$$

By (1), $\varphi Z \subset f\varphi Z$. Conversely, by (2), $f\varphi Z = f \cap \mathscr{Z} \subset \bigcap f\mathscr{Z} \subset \bigcap \mathscr{Z} = \varphi Z$, hence

(3) $$f\varphi Z = \varphi Z.$$

By (2),

(4) $$Z_2 \in \varphi Z_1 \Rightarrow \varphi Z_2 \subset \varphi Z_1.$$

As $\varphi Z \cap \{V \,|\, V \subset Z\}$ is one of the $\mathscr{Z}$'s in (2),

(5) $$V \in \varphi Z \Rightarrow V \subset Z.$$

By (4) and (5), $\varphi Z^+ \subset \varphi Z - \{Z\}$. On the other hand, $\{Z\} \cup \varphi Z^+$ is one of the $\mathscr{Z}$'s in (2), hence

(6) $$\varphi Z^+ = \varphi Z - \{Z\}.$$