tively, i. e. $x_1^* = x_1$, $x_2^* = -x_2$. Moreover, $|x|^2 = |x_1|^2 + |x_2|^2$, $x_1x_2 = x_2x_1$, and there exists one and only one idempotent e such that $z_1^2 = |z_1|^2 e$, and $z_2^2 = -|z_2|^2 e$ for self-adjoint elements z_1 and skew elements z_2.

Hence, by a simple computation we get the equation

$$x \circ x = (x_1 - x_2)(x_1 + x_2) = x_1^2 - x_2^2 + x_1x_2 - x_2x_1$$
$$= (|x_1|^2 + |x_2|^2)e = |x|^2 e,$$

which completes the proof.

COROLLARY 1. *The subalgebra of $\mathscr{K}(A)$ spanned by squares $x \circ x$ ($x \in \mathscr{K}(A)$) is one-dimensional and, consequently, is isomorphic with the real field.*

COROLLARY 2. *For any pair $x, y \in \mathscr{K}(A)$ we have the inequality*

$$|x \circ x + y \circ y| \geqslant |y \circ y|.$$

K. Urbanik has raised the following problem [5]:

If an absolute-valued algebra satisfies the condition $|x^2 + y^2| \geqslant |x^2|$ for all x and y must it be isomorphic with the field of real numbers?

Since the algebra $\mathscr{K}(A)$ may be infinite-dimensional (see [4], p. 252), Corollary 2 gives a negative answer to this problem.

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REMARKS ON ORDERED ABSOLUTE-VALUED ALGEBRAS

BY

K. URBANIK (WROCŁAW)

Let A be a not necessarily associative algebra over the real field R, which is a normed linear space under a norm $|\ |$ satisfying, in addition to the usual requirements, the equality $|xy| = |x|\,|y|$ $(x, y \in A)$. Such an algebra is called *absolute-valued* (see [1], [2], [5], p. 337).

The aim of this note is to study ordered absolute-valued algebras. An *ordering* of an absolute-valued algebra A is determined by the set A^+ of all positive elements of A, i. e. A can be ordered if and only if there exists a subset A^+ of A satisfying the following conditions:

(i) $0 \notin A^+$,

(ii) A^+ is closed with respect to multiplication by positive real numbers and with respect to addition and multiplication in A,

(iii) if $a \neq 0$ and $a \notin A^+$, then $-a \in A^+$.

In fact, one can define $a > b$ if $a - b \in A^+$.

An absolute-valued algebra A is said to be of *real character* if $x^2 + y^2 \neq 0$ and $xy + yx \neq 0$ whenever $x \neq 0$ and $y \neq 0$ $(x, y \in A)$. Obviously, each ordered absolute-valued algebra is of real character. The converse implication is not true. Namely, the following theorem holds:

THEOREM 1. *There exists an absolute-valued algebra of real character, which cannot be ordered.*

Proof. The construction of the algebra satisfying the assertion of the Theorem is similar to that presented in [7], p. 861. Let A_0 be the space of all sequences $x = \{x_n\}$ of real numbers containing only a finite number of non-zero elements. A_0 is a normed space with respect to the norm $|x| = \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2}$ and with the usual addition and scalar multiplication

$$\{x_n\} + \{y_n\} = \{x_n + y_n\}, \quad \lambda\{x_n\} = \{\lambda x_n\}.$$

Let φ be a one-to-one correspondence of the set of all ordered pairs of natural numbers onto the set of all natural numbers satisfying the conditions

(1) $\varphi(1,1) = 2$, $\varphi(2,1) = 1$,

(2) $\varphi(k_1, m_1) \leqslant \varphi(k_2, m_2)$ whenever $k_1 \leqslant k_2$, $m_1 \leqslant m_2$, and either $k_1 \geqslant 2$ or $m_2 \geqslant 2$.

The existence of such a correspondence can be easily shown. Put $\varepsilon_{11} = -1$ and $\varepsilon_{km} = 1$ in the remaining cases. We define the multiplication of elements of A_0 as follows: $\{x_n\}\{y_n\} = \{z_n\}$, where $z_{\varphi(k,m)} = \varepsilon_{km}x_k y_m$ $(k, m = 1, 2, \ldots)$. This product makes A_0 an algebra over the real field. Moreover, A_0 is absolute-valued. Indeed, we have the equation

$$|xy| = \Big(\sum_{n=1}^{\infty} z_n^2\Big)^{1/2} = \Big(\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} z_{\varphi(k,m)}^2\Big)^{1/2} = \Big(\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} x_k^2 y_m^2\Big)^{1/2}$$
$$= \Big(\sum_{k=1}^{\infty} x_k^2\Big)^{1/2}\Big(\sum_{m=1}^{\infty} y_m^2\Big)^{1/2} = |x|\,|y|.$$

Given an element $x = \{x_n\}$ $(x \neq 0)$ of A_0, by $n(x)$ we denote the greatest index n for which $x_n \neq 0$. By virtue of (1) and (2) we get the formula

$$n(xy) = \varphi\big(n(x), n(y)\big) \geqslant 3$$

whenever either $n(x) \geqslant 3$ or $n(y) \geqslant 2$. Moreover, if $1 \leqslant n(x) \leqslant 2$ and $n(y) = 1$, then $n(xy) \leqslant 2$. Hence, by a simple reasoning, we get the equations

$$n(x^2+y^2) = \max\Big(\varphi\big(n(x), n(x)\big), \varphi\big(n(y), n(y)\big)\Big),$$
$$n(xy+yx) = \max\Big(\varphi\big(n(x), n(y)\big), \varphi\big(n(y), n(x)\big)\Big)$$

if either $n(x) \geqslant 2$ or $n(y) \geqslant 2$. Thus in this case we obtain the inequalities $x^2+y^2 \neq 0$ and $xy+yx \neq 0$. In the remaining case $n(x) = n(y) = 1$ we have the formulas $n(x^2+y^2) = 2$ and $n(xy+yx) = 2$, which complete the proof of the real character of A_0.

Put $a = \{a_n\}$, where $a_1 = 1$ and $a_n = 0$ for $n \geqslant 2$. By simple computations we get the equation $(a^2)a = -a$. Suppose that the algebra A_0 can be ordered. Then the element a^2 is positive and, consequently, according to the last equation both a and $-a$ are either positive or negative, which is impossible. The Theorem is thus proved.

Given any subset B of an algebra A, $\dim B$ will denote the linear dimension of B, i. e. the power of a maximal set of linearly independent elements of B. The algebra A_0 of real character constructed in Theorem 1 is infinitely dimensional. Now we shall prove the following theorem:

THEOREM 2. *The real field is the only (up to an isomorphism) finitely dimensional absolute-valued algebra of real character.*

Proof. To prove our theorem it suffices to show that every finitely dimensional absolute-valued algebra of real character is one-dimensional. By Albert's Theorem ([1]) any such algebra A is an isotope of one of the following: the real field, the complex field, the quaternion algebra, or the Cayley algebra. In other words: a new multiplication $x\circ y$ can be introduced in A, such that the algebra A becomes alternative, i. e. the both alternative laws hold,

$$x\circ(x\circ y) = (x\circ x)\circ y, \qquad (x\circ y)\circ y = x\circ(y\circ y),$$

and A has a unit element e. Moreover, the multiplication xy in A is defined in terms of the multiplication $x\circ y$ by the relation

$$xy = (Ux)\circ(Vy), \tag{3}$$

where U and V are fixed invertible linear isometries on A. The norm in A is simply a Euclidean norm.

Let V^{-1} be the inverse of the transformation V, and p^{-1} the inverse in the sense of the multiplication $\circ$ of the element $p = UV^{-1}e$. The formula

$$Wx = p^{-1}\circ(UV^{-1}x) \qquad (x\in A)$$

defines a linear isometry W on A. Since $p\circ(p^{-1}\circ x) = x$ for every $x\in A$, we have, according to (3), the equation

$$xy = (p\circ WVx)\circ Vy \qquad (x, y\in A). \tag{4}$$

First we shall prove that $\dim A \leqslant 2$. Contrary to this let us suppose that $\dim A > 2$. From the definition of the isometry W it follows that $We = e$ and, consequently, e is a proper vector of W. Of course, all proper subspaces of W are either one-dimensional or two-dimensional. Let us assume that W has a two-dimensional proper subspace spanned by an orthonormal basis consisting of b_1 and b_2. Of course, b_1 and b_2 are orthogonal to the unit element e and, consequently, $b_1\circ b_1 = b_2\circ b_2 = -e$. Put

$$Wb_1 = \alpha b_1 + \beta b_2,$$
$$Wb_2 = -\varepsilon\beta b_1 + \varepsilon\alpha b_2,$$

where $\alpha^2+\beta^2 = 1$ and $\varepsilon = 1$ or -1. Setting $a_1 = V^{-1}b_1$, $a_2 = V^{-1}b_2$ we have the equation

$$a_1^2 + a_2^2 = (p\circ Wb_1)\circ b_1 + (p\circ Wb_2)\circ b_2$$
$$= \alpha(p\circ b_1)\circ b_1 + \beta(p\circ b_2)\circ b_1 - \varepsilon\beta(p\circ b_1)\circ b_2 + \varepsilon\alpha(p\circ b_2)\circ b_2.$$

Since the elements e, b_1, b_2, are mutually orthogonal, we have, by the alternativity, $(p\circ b_1)\circ b_2 = -(p\circ b_2)\circ b_1$. Further,

$$(p\circ b_1)\circ b_1 = p\circ(b_1\circ b_1) = -p\circ e = -p,$$

and

$$(p\circ b_2)\circ b_2 = p\circ(b_2\circ b_2) = -p\circ e = -p.$$

Thus

$$a_1^2+a_2^2 = -(1+\varepsilon)\big(ap+\beta(p\circ b_1)\circ b_2\big).$$

Since $a_1^2+a_2^2 \neq 0$, we infer that $\varepsilon = 1$. Now, by simple computations, we obtain the equation

$$a_1a_2+a_2a_1 = (p\circ Wb_1)\circ b_2+(p\circ Wb_2)\circ b_1 = 0,$$

which is impossible. Thus the inequality $\dim A \leqslant 2$ is proved in the case when the transformation W has a two-dimensional proper subspace.

Now let us assume that all proper subspaces of W are one-dimensional. Let e, c_1, c_2 be an orthonormal system of proper vectors of W. By ε_1 and ε_2 we shall denote the proper values of c_1 and c_2, respectively. Of course, $\varepsilon_1 = 1$ or -1, and $\varepsilon_2 = 1$ or -1. Put $d_0 = V^{-1}e$, $d_1 = V^{-1}c_1$ and $d_2 = V^{-1}c_2$. By simple computations we get the equations

$$d_0^2+d_1^2 = (1-\varepsilon_1)p,$$

$$d_0^2+d_2^2 = (1-\varepsilon_2)p,$$

$$d_1d_2+d_2d_1 = (\varepsilon_1-\varepsilon_2)\big((p\circ c_1)\circ c_2\big).$$

Since the left-hand sides of these equations are different from 0, we have the inequalities $\varepsilon_1 \neq 1$, $\varepsilon_2 \neq 1$, and $\varepsilon_1 \neq \varepsilon_2$, which gives a contradiction. This completes the proof of the inequality $\dim A \leqslant 2$.

Suppose that $\dim A = 2$, i. e. that the algebra A is an isotope of the complex field. Then equation (4) can be rewritten in one of the following forms:

$$xy = a\circ(x\circ y), \tag{5}$$

$$xy = a\circ(x^*\circ y^*), \tag{6}$$

$$xy = a\circ(x\circ y^*), \tag{7}$$

$$xy = a\circ(x^*\circ y), \tag{8}$$

where x^* denotes the complex conjugate of x and a is an element with $|a| = 1$. Denoting by i the element orthogonal to e and satisfying the equation $i\circ i = -e$, we have $e^2+i^2 = 0$ in the cases (5) and (6), and $ei+ie = 0$ in the cases (7) and (8), which contradicts the real character of A. Thus $\dim A = 1$, which completes the proof.

Since the real field can be ordered in one and only one way, from Theorem 2 we obtain the following

COROLLARY. *The real field with the natural order is the only (up to an isomorphism) finitely dimensional ordered absolute-valued algebra.*

Now we shall show that the assumption of finite dimension is essential.

THEOREM 3. *There exists an infinitely dimensional ordered absolute-valued algebra.*

Proof. Consider a free groupoid G generated by an element g (see [5], p. 162). Each element a of G different from g can be represented in exactly one way as a product $a = a_1a_2$, where $a_1, a_2 \in G$. We define the length of elements of G inductively. The element g is the only element of length 1. Further, if a_1 and a_2 are of length n_1 and n_2 respectively, then the length of the product a_1a_2 is equal to n_1+n_2. In other words, the length of an element a from G is the number of g's occurring in the expression a. In the sequel $l(a)$ will denote the length of the element a.

Now we shall define by induction a well-ordering of the groupoid G. We assume that $a \prec b$ whenever $l(a) < l(b)$. Suppose that the elements of length less than n are ordered, and $l(a) = l(b) = n$ $(n \geqslant 2)$. The elements a and b can be written in the form a_1a_2 and b_1b_2 respectively. Of course, $l(a_1) < n$, $l(a_2) < n$, $l(b_1) < n$ and $l(b_2) < n$. We put $a \prec b$ if either $a_1 \prec b_1$ or $a_1 = b_1$ and $a_2 \prec b_2$. It is very easy to verify that $ab \prec cd$ whenever either $a \prec c$, $b \preceq d$, or $a \preceq c$, $b \prec d$.

Let A be a real separable Hilbert space and let $a \to i_a$ be a one-to-one correspondence of the groupoid G onto an orthonormal basis of A. The product

$$\Big(\sum_{a\in G}\lambda_a i_a\Big)\Big(\sum_{a\in G}\mu_a i_a\Big) = \sum_{a\in G}\sum_{b\in G}\lambda_a\mu_b i_{ab}$$

makes A an absolute-valued algebra. For each non-zero element $x \in A$ by $\lambda(x)$ we denote the non-zero coefficient λ_a with minimal index a in the expansion $x = \sum_{a\in G}\lambda_a i_a$. Let A be the set of all elements x with $\lambda(x) > 0$. It is easy to verify that the set A fulfils all conditions (i), (ii), and (iii). The Theorem is thus proved.

In the sequel by $S(A)$ we shall denote the set of all squares of elements from the algebra A. Further, $[B]$ will denote the linear set spanned by the elements of B $(B \subset A)$. Now we shall prove a generalization of Theorem 2.

THEOREM 4. *The real field is the only (up to an isomorphism) absolute-valued algebra A of real character for which $S(A)$ is finitely dimensional.*

Proof. Since $xy+yx = (x+y)^2-x^2-y^2$, we infer that $xy+yx \in [S(A)]$ for any $x, y \in A$. Let $e_1, e_2, \ldots, e_n$ be a basis of the linear subspace

$[S(A)]$. The expression $xy+yx$ $(x, y \in A)$ induces n real bilinear functionals $\lambda_1(x, y), \lambda_2(x, y), \ldots, \lambda_n(x, y)$ by means of the expansion

$$xy+yx = \sum_{k=1}^{n} \lambda_k(x, y) e_k .$$

Suppose that $\dim A > \dim S(A)$. Let e_{n+1} be an element of A linearly independent of $e_1, e_2, \ldots, e_n$. For any element $a \in A$ the system of linear equations

$$\sum_{k=1}^{n+1} \beta_k \lambda_j(a, e_k) = 0 \qquad (j = 1, 2, \ldots, n)$$

has a non-zero solution $\beta_1, \beta_2, \ldots, \beta_{n+1}$. Put $b = \sum_{k=1}^{n+1} \beta_k e_k$. Of course, $b \neq 0$ and

$$ab+ba = \sum_{j=1}^{n} \lambda_j(a, b) e_j = \sum_{j=1}^{n} \sum_{k=1}^{n+1} \beta_k \lambda_j(a, e_k) e_j = 0 ,$$

which contradicts the real character of A. Thus, $\dim A = \dim S(A)$ and our Theorem is now a direct consequence of Theorem 2.

COROLLARY. *The real field with the natural order is the only (up to an isomorphism) ordered absolute-valued algebra A for which $S(A)$ is finitely dimensional.*

An operation * defined on an absolute-valued algebra A is called an *involution* if it satisfies the conditions

$$(\lambda x+\mu y)^* = \lambda x^* + \mu y^* ,$$

$$x^{**} = x, \quad xx^* = x^* x, \quad (xy)^* = y^* x^*, \quad |x^*| = |x|$$

for any $\lambda, \mu \in R$, and $x, y \in A$ (see [8]). We say that an absolute-valued algebra A with the multiplication xy is *induced by an involution*, if there exists a new multiplication $x \circ y$ in A making A an absolute-valued algebra such that the multiplication xy is defined in the terms of the multiplication $x \circ y$ by the relation $xy = x^* \circ y$, where * is an involution with respect to the multiplication $\circ$ (see [3]). In [4] for algebras induced by a non-trivial involution the equation $\dim S(A) = 1$ was proved. Now we shall prove the converse implication.

THEOREM 5. *If $\dim S(A) = 1$, then the algebra A is induced by an involution.*

Before proving the Theorem we shall prove a Lemma.

LEMMA. *If $\dim S(A) = 1$, then there exists an idempotent e such that for any $x \in A$ the equation $x^2 = |x|^2 e$ holds.*

Proof. Let a be an arbitrary element of A with

(9) $$|a| = 1 .$$

Setting $e = (a^2)^2$ we have

(10) $$|e| = 1 ,$$

$e \in S(A)$ and, by virtue of the equation $\dim S(A) = 1$, $a^2 = \alpha e$, where $\alpha \in R$. From (9) and (10) it follows that either $\alpha = 1$ or $\alpha = -1$. Thus, $e = (a^2)^2 = (\alpha e)^2 = e^2$. Further, for any element $x \in A$ we have the equation $x^2 = \lambda e$ $(\lambda \in R)$, whence follows the equation $|\lambda| = |x|^2$. Consequently, either $x^2 = |x|^2 e$, or $x^2 = -|x|^2 e$. Contrary to the assertion of the Lemma, let us suppose that there exists an element c different from 0 such that $c^2 = -|c|^2 e$. The function $f(\mu)$ defined by the formula $(c+\mu e)^2 = f(\mu) e$ is continuous. Since

$$f(|c|)e + f(-|c|)e = (c+|c|e)^2 + (c-|c|e)^2 = 2c^2 + 2|c|^2 e = 0 ,$$

we can choose a number ν for which $f(\nu) = 0$. Thus $(c+\nu e)^2 = 0$ and, consequently, $c+\nu e = 0$. Hence we get the equation

$$-|c|^2 e = c^2 = \nu^2 e ,$$

which gives a contradiction. The Lemma is thus proved.

Proof of Theorem 5. Let e be the idempotent satisfying the assertion of the Lemma. From the equation $(x+y)^2 + (x-y)^2 = 2x^2 + 2y^2$ we obtain the inequality

$$|x+y|^2 + |x-y|^2 \geqslant 2|x^2+y^2| = 2\big||x|^2 e + |y|^2 e\big| = 2(|x|^2+|y|^2).$$

Thus, by Schoenberg's Theorem ([6]), A is an inner product space.

Now we shall prove that if u and v in A are mutually orthogonal, then the product uv is orthogonal to e and

(11) $$uv+vu = 0 .$$

Of course, it is sufficient to prove this statement under the additional assumption $|u| = |v| = 1$. Since the transformation $x \to ux$ is an isometry on A and $u^2 = e$, we have the equation

$$(uv, e) = (uv, u^2) = (v, u) = 0 .$$

Thus uv and, consequently, by symmetry, vu, are orthogonal to e. Hence, and from the equations

$$|2e| = 2 = |u+v|^2 = |(u+v)^2| = |u^2+v^2+uv+vu| = |2e+uv+vu| ,$$

$$|2e| = 2 = |u-v|^2 = |(u-v)^2| = |u^2+v^2-uv-vu| = |2e-uv-vu| ,$$

formula (11) follows.

Let A_s be the orthogonal complement in A of the one-dimensional subspace spanned by the element e. Each element x in A can be written in the form $x = \lambda e + x_1$, where $\lambda \in R$ and $x_1 \in A_s$. Put

$$x^* = \lambda e - x_1 \tag{12}$$

and

$$x \circ y = x^* y. \tag{13}$$

Since $|x^*| = |x|$, A remains an absolute-valued algebra with respect to the multiplication (13). Now we shall prove that formula (12) defines an involution with respect to the multiplication (13). The equations $(\lambda x + \mu y)^* = \lambda x^* + \mu y^*$ and $x^{**} = x$ are obvious. Taking into account formula (11) we get the equation

$$\begin{aligned} x \circ x^* &= (x^*)^2 = (\lambda e - x_1)^2 = \lambda^2 e + x_1^2 - \lambda(ex_1 + x_1 e) \\ &= \lambda^2 e + x_1^2 = \lambda^2 e + x_1^2 + \lambda(ex_1 + x_1 e) = (\lambda e + x_1)^2 = x^2 = x^* \circ x. \end{aligned}$$

Put $x = \lambda e + x_1$, $y = \alpha e + \beta x_1 + x_2$, where $x_1, x_2 \in A_s$ and x_2 is orthogonal to x_1. Since ex_1, $x_1 e$, ex_2 and $x_1 x_2$ are orthogonal to e, we have, in view of (11),

$$\begin{aligned} (x \circ y)^* = (x^* y)^* &= \big((\lambda e - x_1)(\alpha e + \beta x_1 + x_2)\big)^* \\ &= (\alpha\lambda e - \beta|x_1|^2 e - \alpha x_1 e + \beta\lambda e x_1 + \lambda e x_2 - x_1 x_2)^* \\ &= (\alpha\lambda - \beta|x_1|^2)e + \alpha x_1 e - \beta\lambda e x_1 - \lambda e x_2 + x_1 x_2 \\ &= (\alpha\lambda - \beta|x_1|^2)e - \alpha e x_1 + \beta\lambda x_1 e + \lambda x_2 e - x_2 x_1 \\ &= (\alpha e + \beta x_1 + x_2)(\lambda e - x_1) = y x^* = y^* \circ x^*. \end{aligned}$$

Thus the transformation (12) is an involution. Finally, the multiplication xy is defined in the terms of the multiplication (13) by the relation $xy = x^* \circ y$, which completes the proof.

From Theorems 2 and 5 we get the following

COROLLARY. *If the subalgebra generated by squares of elements of an absolute-valued algebra A is of real character and finitely dimensional, then the algebra A is induced by an involution.*

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INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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