An example of an $(\mathfrak{m},\mathfrak{n})$-product can be constructed as follows: Let X_t be the Stone space of $\mathfrak{A}_t$, let g_t be the Stone isomorphism of $\mathfrak{A}_t$ onto the field of all clopen subsets of X_t, and let X be the Cartesian product of all the spaces X_t. For every $A\in\mathfrak{A}_t$, let $g_t^*(A)$ = the set of all points in X whose t^{th} coordinate is in $g_t(A)$.

Let $\mathfrak{F}$ be the smallest field (of subsets of X) containing all the intersections $\bigcap_{t\in T'} g_t^*(A_t)$, where $A_t\in\mathfrak{A}_t$ and $T'\subset T$, $\overline{\overline{T'}}\leqslant\mathfrak{n}$. Finally, let $(i,\mathfrak{B})$ be any $\mathfrak{m}$-extension of the Boolean algebra $\mathfrak{F}$. Then

$$((ig_t^*)_{t\in T},\ \mathfrak{B}) \tag{**}$$

is an $(\mathfrak{m},\mathfrak{n})$-product of $(\mathfrak{A}_t)_{t\in T}$.

Problem 9. Is every $(\mathfrak{m},\mathfrak{n})$-product of $(\mathfrak{A}_t)_{t\in T}$ of the form (**)? (**P 441**)

I should like also to recall that my problem on principal ideals in the field of all subsets of a set (Sikorski [5], P 61) is not yet solved.

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COLLOQUIUM MATHEMATICUM

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A REMARK ON ABSOLUTE-VALUED ALGEBRAS

BY

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An algebra A over the real field R is called *absolute-valued* if it is a normed space under a multiplicative norm $|\ |$, i. e. a norm satisfying, in addition to the usual requirements, the condition $|xy| = |x|\,|y|$ for all $x, y\in A$ (see [1]).

An operation * defined on A is called an *involution* if it satisfies the following conditions:

$$(\lambda x+\mu y)^* = \lambda x^*+\mu y^*,$$

$$x^{**} = x,\qquad xx^* = x^*x,\qquad (xy)^* = y^*x^*,\qquad |x^*| = |x|$$

for any $\lambda,\mu\in R$ and $x, y\in A$ (see [4]).

We say that an involution is *non-trivial* if it is different from the identity operation.

In every absolute-valued algebra A with an involution we can introduce a new multiplication by means of the formula

$$x\circ y = x^*y.$$

The algebra A with this product will be denoted by $\mathscr{K}(A)$. $\mathscr{K}(A)$ remains an absolute-valued algebra. The algebra $\mathscr{K}(A)$ is called a *cracovian algebra generated by* A or an *algebra induced by involution* (see [2], [3]).

THEOREM. *If A is an absolute-valued algebra with a non-trivial involution, then there exists in $\mathscr{K}(A)$ an element e such that*

$$x\circ x = |x|^2 e$$

for any $x\in\mathscr{K}(A)$.

Proof. Using the well-known process of embedding linear normed spaces in Banach spaces, we can prove that the algebra A can be extended to a complete algebra. Thus, without loss of generality, we may assume that the algebra A is complete. For complete algebras it was proved in [4] that each element $x\in A$ can be represented as a sum $x = x_1+x_2$, where the elements x_1 and x_2 are self-adjoint and skew respectively, i. e. $x_1^* = x_1$, $x_2^* = -x_2$. Moreover, $|x|^2 = |x_1|^2 + |x_2|^2$, $x_1x_2 = x_2x_1$, and there exists one and only one idempotent e such that $z_1^2 = |z_1|^2e$, and $z_2^2 = -|z_2|^2e$ for self-adjoint elements z_1 and skew elements z_2.

Hence, by a simple computation we get the equation

$$x \circ x = (x_1 - x_2)(x_1 + x_2) = x_1^2 - x_2^2 + x_1x_2 - x_2x_1$$
$$= (|x_1|^2 + |x_2|^2)e = |x|^2e,$$

which completes the proof.

COROLLARY 1. *The subalgebra of $\mathcal{K}(A)$ spanned by squares $x \circ x$ ($x \in \mathcal{K}(A)$) is one-dimensional and, consequently, is isomorphic with the real field.*

COROLLARY 2. *For any pair $x, y \in \mathcal{K}(A)$ we have the inequality*

$$|x \circ x + y \circ y| \geqslant |y \circ y|.$$

K. Urbanik has raised the following problem [5]:

If an absolute-valued algebra satisfies the condition $|x^2 + y^2| \geqslant |x^2|$ for all x and y must it be isomorphic with the field of real numbers?

Since the algebra $\mathcal{K}(A)$ may be infinite-dimensional (see [4], p. 252), Corollary 2 gives a negative answer to this problem.

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REMARKS ON ORDERED ABSOLUTE-VALUED ALGEBRAS

BY

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Let A be a not necessarily associative algebra over the real field R, which is a normed linear space under a norm $|\ |$ satisfying, in addition to the usual requirements, the equality $|xy| = |x|\,|y|$ $(x, y \in A)$. Such an algebra is called *absolute-valued* (see [1], [2], [5], p. 337).

The aim of this note is to study ordered absolute-valued algebras. An *ordering* of an absolute-valued algebra A is determined by the set A^+ of all positive elements of A, i. e. A can be ordered if and only if there exists a subset A^+ of A satisfying the following conditions:

(i) $0 \notin A^+$,

(ii) A^+ is closed with respect to multiplication by positive real numbers and with respect to addition and multiplication in A,

(iii) if $a \neq 0$ and $a \notin A^+$, then $-a \in A^+$.

In fact, one can define $a > b$ if $a - b \in A^+$.

An absolute-valued algebra A is said to be of *real character* if $x^2 + y^2 \neq 0$ and $xy + yx \neq 0$ whenever $x \neq 0$ and $y \neq 0$ $(x, y \in A)$. Obviously, each ordered absolute-valued algebra is of real character. The converse implication is not true. Namely, the following theorem holds:

THEOREM 1. *There exists an absolute-valued algebra of real character, which cannot be ordered.*

Proof. The construction of the algebra satisfying the assertion of the Theorem is similar to that presented in [7], p. 861. Let A_0 be the space of all sequences $x = \{x_n\}$ of real numbers containing only a finite number of non-zero elements. A_0 is a normed space with respect to the norm $|x| = (\sum_{n=1}^{\infty} x_n^2)^{1/2}$ and with the usual addition and scalar multiplication

$$\{x_n\} + \{y_n\} = \{x_n + y_n\}, \quad \lambda\{x_n\} = \{\lambda x_n\}.$$

Let φ be a one-to-one correspondence of the set of all ordered pairs of natural numbers onto the set of all natural numbers satisfying the