For every subset $W \subset X$, $\overline{\overline{W}} \leqslant \mathfrak{m}$, and for every $\varphi \in Z$, let $L_{W,\varphi}$ be a subset of $Z$ defined as follows:

$$f \in L_{W,\varphi} \text{ if and only if } f|W = \varphi|W$$

(i. e. if $f(x) = \varphi(x)$ for every $x \in W$).

$F$ is the $\mathfrak{m}$-field of subsets of $Z$ generated by all the sets $L_{W,\varphi}$.

Now we are going to prove that $F^{\mathfrak{m}}$ is not weakly $\mathfrak{m}$-distributive, if $\mathfrak{n} \geqslant \mathfrak{m}^+$.

Let

$$T(x, \eta) = \{f \in Z : f(x) = \eta\},$$

and let

$$A_\eta = \{T(x, \eta) : x \in X\}.$$

Every family $A_\eta$, $\eta \in Y$, $\mathfrak{m}^+$-covers $F$ (see [2], p. 139).

It follows from the definition of $Z$ that

(i) $$\bigcap_{\eta \in Y} \bigcup_{x \in X} T(x, \eta) = 0,$$

where the intersection and the union are set-theoretical.

Suppose that there exists a covering $A$ of $F$ which $\mathfrak{m}$-refines every $A_\eta$. Thus, the field $F$ being $\mathfrak{m}$-complete, every element of $A$ is included in the set-theoretical union of elements of $A_\eta$, for every $\eta \in Y$. Therefore it is empty, by (i). Contradiction.

Consequently, by lemma 2.5, $F^{\mathfrak{m}}$ is not weakly $\mathfrak{m}$-distributive.

## REFERENCES


[1] S. Banach et C. Kuratowski, *Sur une généralisation du problème de la mesure*, Fundamenta Mathematicae 14 (1929), p. 127-131.

[2] R. S. Pierce, *Distributivity and the normal completion of Boolean algebras*, Pacific Journal of Mathematics 8 (1958), p. 133-140.

[3] R. Sikorski, *Boolean algebras*, Berlin-Göttingen-Heidelberg 1960.


*Reçu par la Rédaction le 8. 8. 1962*

COLLOQUIUM MATHEMATICUM

VOL. XI 1963 FASC. 1

# *A FEW PROBLEMS ON BOOLEAN ALGEBRAS*

BY

ROMAN SIKORSKI (WARSAW)

The purpose of this short note is to collect a few problems concerning Boolean algebras which seem to be interesting. Some of them were mentioned in my expository paper [7], others were quoted in my book [9]. Perhaps, the level of difficulty of some of them is rather low. In any case, their solutions will mean a progress in the theory of Boolean algebras.

The first problem concerns the following simple theorem: If a Boolean algebra $\mathfrak{A}$ is $\mathfrak{m}'$-complete for every infinite cardinal $\mathfrak{m}' < \mathfrak{m}$, $A, A_t \in \mathfrak{A}$, $A = \bigcup_{t \in T} A_t$ and $\overline{\overline{T}} \leqslant \mathfrak{m}$, then there exist elements $B_t \in \mathfrak{A}$ $(t \in T)$ such that

$$B_t \subset A_t, \quad B_t \cap B_{t'} = 0 \quad \text{for} \quad t \neq t' \text{ and } A = \bigcup_{t \in T} B_t.$$

Problem 1. Is this theorem true without the hypothesis that $\mathfrak{A}$ is $\mathfrak{m}'$-complete for every $\mathfrak{m}' < \mathfrak{m}$? (**P 434**).

Another problem of this kind is

Problem 2. Find, for every uncountable cardinal $\mathfrak{m}$, a Boolean $\mathfrak{m}$-algebra $\mathfrak{A}$ with the property: if the join $\bigcup_{t \in T} A_t$ exists in $\mathfrak{A}$ and $\overline{\overline{T}} \leqslant \mathfrak{m}$, then there exists a finite subset $T' \subset T$ such that $\bigcup_{t \in T} A_t = \bigcup_{t \in T'} A_t$. (**P 435**).

For $\mathfrak{m} = \aleph_0$ an example of such a Boolean algebra was given by Sierpiński [4].

Problems 3-6 which follow are connected with a classification of Boolean algebras discussed in my paper [7].

Problem 3. Find an example (for every uncountable cardinal $\mathfrak{m}$) of a weakly $\mathfrak{m}$-distributive Boolean $\mathfrak{m}$-algebra which is not $\mathfrak{m}$-distributive (**P 436**).

In the case where $\mathfrak{m} = \aleph_0$ such an example is given by non-atomic measure algebras (i. e. Boolean $\sigma$-algebras with a strictly positive finite $\sigma$-measure). Other examples can be obtained e. g. by forming the direct union or the Boolean product of a non-atomic measure algebra and a $\sigma$-distributive $\sigma$-algebra. Thus the following problem arises:

Problem 4. Construct an example of a non-$\sigma$-distributive but weakly $\sigma$-distributive Boolean $\sigma$-algebra without using measure algebras (**P 437**).

Let $\mathfrak{S}$ be a set of $\mathfrak{m}$-generators of a Boolean $\mathfrak{m}$-algebra $\mathfrak{A}$, and let $f$ be a mapping from $\mathfrak{S}$ into a Boolean $\mathfrak{m}$-algebra $\mathfrak{B}$. If $f$ can be extended to an $\mathfrak{m}$-homomorphism $h$ from $\mathfrak{A}$ into $\mathfrak{B}$, then $f$ satisfies the following condition:

(a) if $\bigcap_{t\epsilon T}^{\mathfrak{A}}\eta(t)\cdot A_t = 0$, where $\eta(t) = \pm 1$, $A_t \epsilon \mathfrak{S}$ and $\overline{\overline{T}} < \mathfrak{m}$, then $\bigcap_{t\epsilon T}^{\mathfrak{B}}\eta(t)\cdot f(A_t) = 0$.

Here the convention

$$+1\cdot A = A \text{ and } -1\cdot A = -A = \text{the complement of } A$$

is assumed.

The necessary condition (a) for $f$ to have an extension to an $\mathfrak{m}$-homomorphism is not sufficient, in general. A Boolean $\mathfrak{m}$-algebra $\mathfrak{B}$ is said to have the *strong* $\mathfrak{m}$-*extension property* if every mapping $f$ from a set $\mathfrak{S}$ of $\mathfrak{m}$-generators of an $\mathfrak{m}$-algebra $\mathfrak{A}$ into $\mathfrak{B}$, such that (a) holds, can be extended to an $\mathfrak{m}$-homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$. It is known (see Sikorski [6], theorem 34,1) that every $\mathfrak{m}$-field of sets has the strong $\mathfrak{m}$-extension property. Hence it follows easily that every $\mathfrak{m}$-distributive $\mathfrak{m}$-algebra has the strong $\mathfrak{m}$-extension property (Sikorski [8], Sikorski and Traczyk [10]).

Problem 5. Is every $\mathfrak{m}$-algebra with the strong $\mathfrak{m}$-extension property necessarily $\mathfrak{m}$-distributive? (**P 438**).

Consider now the particular case where the set $\mathfrak{S}$ of $\mathfrak{m}$-generators of $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}$. Then condition (a) implies that $f$ is a homomorphism. Clearly a homomorphism $f$ from $\mathfrak{S}$ into $\mathfrak{B}$ satisfies condition (a) if and only if it satisfies the following condition:

(a′) if $\bigcap_{t\epsilon T}^{\mathfrak{A}} A_t = 0$ where $A_t \epsilon \mathfrak{S}$ and $\overline{\overline{T}} < \mathfrak{m}$, then $\bigcap_{t\epsilon T}^{\mathfrak{B}} f(A_t) = 0$.

Condition (a′) is necessary for the existence of an $\mathfrak{m}$-homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ which is an extension of $f$. However, it is not sufficient. A Boolean $\mathfrak{m}$-algebra $\mathfrak{B}$ is said to have the *weak* $\mathfrak{m}$-*extension property* if every homomorphism $f$ from a subalgebra $\mathfrak{S}$ of a Boolean $\mathfrak{m}$-algebra $\mathfrak{A}$ into $\mathfrak{B}$, such that

(i) $\mathfrak{S}$ $\mathfrak{m}$-generates $\mathfrak{A}$,

(ii) $f$ satisfies (a′),

can be extended to an $\mathfrak{m}$-homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$.

Clearly the strong $\mathfrak{m}$-extension property implies the weak $\mathfrak{m}$-extension property (therefore every $\mathfrak{m}$-distributive $\mathfrak{m}$-algebra has the weak $\mathfrak{m}$-extension property) but they are not equivalent. Dubins [1] proved



that every measure algebra has the weak $\aleph_0$-extension property. Matthes [2] proved a more general theorem: every weakly $\mathfrak{m}$-distributive Boolean $\mathfrak{m}$-algebra has the weak $\mathfrak{m}$-extension property.

Problem 6. Is every Boolean $\mathfrak{m}$-algebra with the weak $\mathfrak{m}$-extension property necessarily weakly $\mathfrak{m}$-distributive? (**P 439**).

It is known that it is always $\mathfrak{m}$-representable (see Sikorski [9]).

Pierce [3] proved that the $\mathfrak{m}$-completion of any $\mathfrak{m}$-distributive Boolean algebra is $\mathfrak{m}$-distributive. We mention here the analogous problem of Traczyk [11]:

Problem 7. Is the $\mathfrak{m}$-completion of a weakly $\mathfrak{m}$-distributive Boolean algebra also weakly $\mathfrak{m}$-distributive? (**P 433**).

Traczyk [11] proved that the answer is affirmative if the algebra in question satisfies the $\mathfrak{m}$-chain condition.

A pair $(i, \mathfrak{B})$ is said to be an $\mathfrak{m}$-*extension* of a Boolean algebra $\mathfrak{A}$ provided

($e_1$) $\mathfrak{B}$ is a Boolean $\mathfrak{m}$-algebra,

($e_2$) $i$ is an $\mathfrak{m}$-isomorphism from $\mathfrak{A}$ into $\mathfrak{B}$,

($e_3$) $i(\mathfrak{A})$ $\mathfrak{m}$-generates $\mathfrak{B}$.

If $(i', \mathfrak{B}')$ is another $\mathfrak{m}$-extension of $\mathfrak{A}$, we write

$$(*) \qquad (i, \mathfrak{B}) \leqslant (i', \mathfrak{B}')$$

if there exists an $\mathfrak{m}$-homomorphism $h$ from $\mathfrak{B}'$ onto (= into) $\mathfrak{B}$ such that $i = hi'$.

Problem 8. Suppose $(i, \mathfrak{B})$ is an $\mathfrak{m}$-completion of $\mathfrak{A}$. Does inequality (*) hold for every $\mathfrak{m}$-extension $(i', \mathfrak{B}')$ of $\mathfrak{A}$? (**P 440**).

This problem was published by me for $\mathfrak{m} = \aleph_0$ in a purely topological formulation in Colloquium Mathematicum 2 (1951), p. 151, P 77.

We know only that if $(i, \mathfrak{B})$ is an $\mathfrak{m}$-completion of $\mathfrak{A}$, then $(i', \mathfrak{B}') \leqslant (i, \mathfrak{B})$ never holds except in the trivial case where $(i', \mathfrak{B}')$ is isomorphic to $(i, \mathfrak{B})$ (i.e. there exists an isomorphism $h$ from $\mathfrak{B}'$ onto $\mathfrak{B}$ such that $i = hi'$).

The affirmative solution of Problem 8 will solve automatically a few similar problems concerning minimal products of Boolean algebras (for details, see Sikorski [8] and Sikorski [9]).

Let $\mathfrak{m}$ and $\mathfrak{n}$ be infinite cardinals, $\mathfrak{n} \leqslant \mathfrak{m}$.

A pair $\big((i_t)_{t\epsilon T}, \mathfrak{B}\big)$ is said to be a *Boolean* $(\mathfrak{m}, \mathfrak{n})$-*product* of an indexed set $(\mathfrak{A}_t)_{t\epsilon T}$ of non-degenerate Boolean algebras provided

($p_1$) $\mathfrak{B}$ is a Boolean $\mathfrak{m}$-algebra,

($p_2$) $i_t$ is an $\mathfrak{m}$-isomorphism from $\mathfrak{A}_t$ into $\mathfrak{B}$,

($p_3$) the union of all the subalgebras $i_t(\mathfrak{A}_t)$ $\mathfrak{m}$-generates $\mathfrak{B}$,

($p_4$) the subalgebras $i_t(\mathfrak{A}_t)$ are $\mathfrak{n}$-independent, i. e.

$$\bigcap_{t\epsilon T'}^{\mathfrak{B}} i_t(A_t) \neq 0 \quad \text{for} \quad A_t \neq 0,\ A_t \epsilon \mathfrak{A}_t,\ \overline{\overline{T'}} \leqslant \mathfrak{n},\ T' \subset T.$$

An example of an $(\mathfrak{m}, \mathfrak{n})$-product can be constructed as follows: Let $X_t$ be the Stone space of $\mathfrak{A}_t$, let $g_t$ be the Stone isomorphism of $\mathfrak{A}_t$ onto the field of all clopen subsets of $X_t$, and let $X$ be the Cartesian product of all the spaces $X_t$. For every $A \in \mathfrak{A}_t$, let $g_t^*(A)$ = the set of all points in $X$ whose $t^{\text{th}}$ coordinate is in $g_t(A)$.

Let $\mathfrak{F}$ be the smallest field (of subsets of $X$) containing all the intersections $\bigcap_{t \in T'} g_t^*(A_t)$, where $A_t \in \mathfrak{A}_t$ and $T' \subset T$, $\overline{\overline{T'}} \leqslant \mathfrak{n}$. Finally, let $(i, \mathfrak{B})$ be any $\mathfrak{m}$-extension of the Boolean algebra $\mathfrak{F}$. Then

$$(**) \qquad \big((ig_t^*)_{t \in T}, \mathfrak{B}\big)$$

is an $(\mathfrak{m}, \mathfrak{n})$-product of $(\mathfrak{A}_t)_{t \in T}$.

Problem 9. Is every $(\mathfrak{m}, \mathfrak{n})$-product of $(\mathfrak{A}_t)_{t \in T}$ of the form $(**)$? (**P 441**)

I should like also to recall that my problem on principal ideals in the field of all subsets of a set (Sikorski [5], P 61) is not yet solved.

### REFERENCES

[1] L. E. Dubins, *Generalized random variables*, Transactions of the American Mathematical Society 84 (1957), p. 273-309.

[2] K. Matthes, *Über die Ausdehnung von* $\aleph$*-Homomorphismen Boolescher Algebren*, Zeitschrift für Mathematik, Logik und Grundlagen der Mathematik 6 (1960), p. 97-105; (*II*) 7 (1961), p. 16-19.

[3] R. S. Pierce, *Distributivity and the normal completion of Boolean algebras*, Pacific Journal of Mathematics 8 (1958), p. 113-140.

[4] W. Sierpiński, *Sur les ensembles presque contenus les uns dans les autres*, Fundamenta Mathematicae 35 (1948), p. 141-150.

[5] R. Sikorski, *On an unsolved problem from the theory of Boolean algebras*, Colloquium Mathematicum 2 (1949), p. 27-29.

[6] — *Boolean algebras*, Berlin-Göttingen-Heidelberg 1960.

[7] — *Representation and distributivity of Boolean algebras*, Colloquium Mathematicum 8 (1961), p. 1-13.

[8] — *On extensions and products of Boolean algebras*, Fundamenta Mathematicae 53 (1963), p. 99-116.

[9] — *Boolean algebras*, second edition, in print.

[10] — and T. Traczyk, *On free products of* $\mathfrak{m}$*-distributive Boolean algebras*, Colloquium Mathematicum 11 (1963), p. 13-16.

[11] T. Traczyk, *Minimal extensions of weakly distributive Boolean algebras*, ibidem 11 (1963), p. 17-24.

*Reçu par la Rédaction le 26. 11. 1962*

COLLOQUIUM MATHEMATICUM

VOL. XI 1963 FASC. 1

# A REMARK ON ABSOLUTE-VALUED ALGEBRAS

BY

B. GLEICHGEWICHT (WROCŁAW)

An algebra $A$ over the real field $R$ is called *absolute-valued* if it is a normed space under a multiplicative norm $|\ |$, i. e. a norm satisfying, in addition to the usual requirements, the condition $|xy| = |x|\,|y|$ for all $x, y \in A$ (see [1]).

An operation $*$ defined on $A$ is called an *involution* if it satisfies the following conditions:

$$(\lambda x + \mu y)^* = \lambda x^* + \mu y^*,$$

$$x^{**} = x, \qquad xx^* = x^*x, \qquad (xy)^* = y^*x^*, \qquad |x^*| = |x|$$

for any $\lambda, \mu \in R$ and $x, y \in A$ (see [4]).

We say that an involution is *non-trivial* if it is different from the identity operation.

In every absolute-valued algebra $A$ with an involution we can introduce a new multiplication by means of the formula

$$x \circ y = x^* y.$$

The algebra $A$ with this product will be denoted by $\mathscr{K}(A)$. $\mathscr{K}(A)$ remains an absolute-valued algebra. The algebra $\mathscr{K}(A)$ is called a *cracovian algebra generated by* $A$ or an *algebra induced by involution* (see [2], [3]).

THEOREM. *If $A$ is an absolute-valued algebra with a non-trivial involution, then there exists in $\mathscr{K}(A)$ an element $e$ such that*

$$x \circ x = |x|^2 e$$

*for any $x \in \mathscr{K}(A)$.*

Proof. Using the well-known process of embedding linear normed spaces in Banach spaces, we can prove that the algebra $A$ can be extended to a complete algebra. Thus, without loss of generality, we may assume that the algebra $A$ is complete. For complete algebras it was proved in [4] that each element $x \in A$ can be represented as a sum $x = x_1 + x_2$, where the elements $x_1$ and $x_2$ are self-adjoint and skew respec-