[16] — *Sur certaines suites liées aux ensembles plans et leur application à la représentation conforme*, ibidem 4 (1957), p. 8-13.

[17] — *Teoria funkcji analitycznych*, Warszawa 1957 (in Polish).

[18] — et Z. Opial, *Un lemme sur les polynômes de Lagrange*, Annales Polonici Mathematici 2 (1955), p. 73-76.

[19] T. Radó, *Subharmonic functions*, New York, 1949.

[20] J. Siciak, *Sur la distribution des points extrémaux dans les ensembles plans*, Annales Polonici Mathematici 4 (1958), p. 214-219.

[21] — *O rozkładzie punktów ekstremalnych odpowiadających funkcjom schodkowym w pewnej klasie zbiorów płaskich*, Zeszyty Naukowe Uniwersytetu Jagiellońskiego 4 (1958), p. 59-63 (in Polish).

[22] — *On some extremal functions and their applications in the theory of functions of several complex variables*, Technical Report, Stanford 1961 (also Transactions of the American Mathematical Society 105 (1962), p. 322-357).

[23] J. L. Walsh, *Interpolation and approximation*, New York 1935.

JAGELLONIAN UNIVERSITY

Reçu par la Rédaction le 15. 3. 1963

COLLOQUIUM MATHEMATICUM

VOL. XI 1964 FASC. 2

ON SOME EXTREMAL FUNCTIONS OF LEJA IN THE SPACE

BY

W. BACH (CRACOW)

Let R^m be m-dimensional Euclidean space, $m \geqslant 2$, E a closed and bounded set in R^m and

$$\omega(p,q) \overset{\mathrm{df}}{=} \begin{cases} |p-q| & \text{for} \quad m=2, \\ e^{-|p-q|^{2-k}} & \text{for} \quad m \geqslant 3. \end{cases}$$

Let $q^{(n)} = \{q_0, q_1, \ldots, q_n\}$ be an n-th extremal system of E with respect to $\omega(p,q)$ (see [2, 4]) and $p^{(n)} = \{p_0, p_1, \ldots, p_n\}$ an arbitrary system of $n+1$ different points of E. We put

$$A_n(r) \overset{\mathrm{df}}{=} \max_{(j)} \prod_{\substack{k=0\\k\neq j}}^{n} \frac{\omega(r,q_k)}{\omega(q_j,q_k)}, \qquad B_n(r) \overset{\mathrm{df}}{=} \inf_{p^{(n)}\subset E} \Big\{ \max_{(j)} \prod_{\substack{k=0\\k\neq j}}^{n} \frac{\omega(r,p_j)}{\omega(p_k,p_j)} \Big\}.$$

It is known [2, 4, 5] that if $m = 2$ and $r \notin E$, then

$$\lim_{n\to\infty} \frac{1}{n} \log A_n(r) = G(r), \tag{1}$$

$$\lim_{n\to\infty} \frac{1}{n} \log B_n(r) = G(r), \tag{2}$$

where $G(r) = I - u(r)$ (see below).

Let D_∞ be the component of the complement of E containing the point $r = \infty$ and F_∞ the boundary of D_∞. If $m = 2$ and $r \in D_\infty$, then $G(r)$ is the Green function for D_∞ with a pole at infinity and for $r \notin D_\infty$ we have $G(r) = 0$ excepting a set of capacity zero.

The object of this paper is to prove (1) and (2) in general case (for $m \geqslant 2$).

Denote by M the class of all positive Radon measures ν such that $\nu(E) = 1$ and $\nu(e \cap E) = 0$ if $e \cap E = \emptyset$.

Let μ be the measure of M such that

$$I \stackrel{\text{df}}{=} \int_E \int_E \log \frac{1}{\omega(p,q)} d\mu(p) d\mu(q) = \inf_{\nu \in M} \int_E \int_E \log \frac{1}{\omega(p,q)} d\nu(p) d\nu(q).$$

The number e^{-I} is the capacity of E. It is known that

$$u(p) \stackrel{\text{df}}{=} \int_E \log \frac{1}{\omega(p,q)} d\mu(q) = I \tag{3}$$

for $p \in E$ excepting a set of capacity zero.

Let $\mu_n(e) = k/(n+1)$, where k is the number of points of $q^{(n)}$ contained in e. Since the measure μ is unique [1, 3], the sequence μ_n is convergent to μ. Hence we obtain (1), because the Fekete's radius and the capacity are equal.

Proof of (2). Let $\bar{\mu}$ be the measure of M such that

$$\bar{I} \stackrel{\text{df}}{=} \int_E \int_E \log \frac{1}{\omega(p,q;r)} d\bar{\mu}(p) d\bar{\mu}(q) = \inf \int_E \int_E \log \frac{1}{\omega(p,q;r)} d\nu(p) d\nu(q)$$

where $\omega(p,q;r) = \omega(p,q)/\omega(p,r)\omega(r,q)$. Using the same method as in [3] (see also [2]) we can easily prove that

$$\bar{u}(p) \stackrel{\text{df}}{=} \int_E \log \frac{1}{\omega(p,q;r)} d\bar{\mu}(q) = \bar{I} \tag{4}$$

for $p \in E$ excepting a set of capacity zero. Integrating (4) with respect to μ we obtain

$$G(r) = \bar{I} + \int_E \log \frac{1}{\omega(q,r)} d\bar{\mu}(q). \tag{5}$$

Let $\bar{q}^{(n)} = \{\bar{q}_0, \bar{q}_1, \ldots, \bar{q}_n\}$ be an n-th extremal system of E with respect to $\omega(p,q;r)$. Without loss of generality we can assume that, for $j = 1, \ldots, n$,

$$\prod_{k=1}^{n} \omega(\bar{q}_0, \bar{q}_k) \leqslant \prod_{\substack{k=0 \\ k \neq j}}^{n} \omega(\bar{q}_j, \bar{q}_k).$$

Using the same method as in the case of $m = 2$ we can easily prove that the Fekete's radius of E with respect to $\omega(p,q;r)$ is equal to the capacity of E with respect to $\omega(p,q;r)$. Hence

$$\lim_{n\to\infty} \Big[\prod_{k=1}^{n} \omega(\bar{q}_0, \bar{q}_k)\Big]^{1/n} = e^{-\bar{I}}. \tag{6}$$

Let $\bar{\mu}_n$ be the measure defined similarly as μ_n, but for the system $\bar{q}^{(n)}$. Since the measure $\bar{\mu}$ is unique (the uniqueness can be proved by the same method as for μ) we have $\lim_{n\to\infty} \bar{\mu}_n = \bar{\mu}$. Hence, for $r \notin E$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log \frac{1}{\omega(r, \bar{q}_k)} \tag{7}$$

$$= \lim_{n\to\infty} \int_E \log \frac{1}{\omega(r,q)} d\bar{\mu}_n(q) = \int_E \log \frac{1}{\omega(p,q)} d\bar{\mu}(q).$$

From (5), (6) and (7) we have

$$\lim_{n\to\infty} \Big\{\max_{(j)} \Big[\frac{1}{n} \sum_{\substack{k=0 \\ k\neq j}}^{n} \log \frac{1}{\omega(\bar{q}_j, \bar{q}_k)} - \log \frac{1}{\omega(\bar{q}_j, r)}\Big]\Big\}$$

$$= \bar{I} + \int_E \log \frac{1}{\omega(q,r)} d\bar{\mu}(q) = G(r).$$

Hence and from the definition of $B_n(r)$ we obtain

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log B_n(r) \leqslant G(r). \tag{8}$$

Let $\{n_k\}$ be a sequence such that

$$\lim_{k\to\infty} \frac{1}{n_k} \log B_{n_k}(r) = \underline{\lim_{n\to\infty}} \frac{1}{n} \log B_n(r).$$

Let $\bar{\bar{q}}^{(n)} = \{\bar{\bar{q}}_0, \bar{\bar{q}}_1, \ldots, \bar{\bar{q}}_n\}$ be an n-th system of points realising $B_n(r)$ and $\bar{\bar{\mu}}_n$ the measure defined with respect to $\bar{\bar{q}}^{(n)}$ analogously as was μ_n with respect to $q^{(n)}$. Without loss of generality we may assume that the sequence $\{n_k\}$ is such that the subsequence $\bar{\bar{\mu}}_{n_k}$ is convergent, say: $\bar{\bar{\mu}}_{n_k} \to \bar{\bar{\mu}}$. Writing

$$\omega_\delta(p,q) \stackrel{\text{df}}{=} \begin{cases} \omega(p,q), & \text{if } \omega(p,q) \geqslant \delta, \\ \delta, & \text{if } \omega(p,q) < \delta, \end{cases}$$

we have for every sufficiently small $\delta > 0$ and p belonging to the support of $\bar{\bar{\mu}}$

$$\lim_{k\to\infty} \frac{1}{n_k} \log B_{n_k}(r) \geqslant \int_E \log \frac{1}{\omega_\delta(p,q)} d\bar{\bar{\mu}}(q) - \log \frac{1}{\omega_\delta(p,q)}$$

and hence

$$\lim_{k\to\infty}\frac{1}{n_k}\log B_{n_k}(r)\geqslant\int\limits_E\log\frac{1}{\omega(p,q)}\,d\bar{\bar{\mu}}(q)-\log\frac{1}{\omega(p,r)}.$$

Since both the measure μ of a set of capacity zero and the measure $\bar{\bar{\mu}}$ of a single point are equal to zero, integrating the last inequality with respect to μ we have, by (3),

$$\lim_{k\to\infty}\frac{1}{n_k}\log B_{n_k}(r)\geqslant\int\limits_{\dot E}\Big[\int\limits_{E}\log\frac{1}{\omega(p,q)}\,d\bar{\bar{\mu}}(q)-\log\frac{1}{\omega(p,r)}\Big]\,d\mu(p)$$

$$=\int\limits_{\dot E}\Big[\int\limits_{E}\log\frac{1}{\omega(p,q)}\,d\mu(p)-\int\limits_{E}\log\frac{1}{\omega(p,r)}\,d\mu(p)\Big]\,d\bar{\bar{\mu}}(q)$$

$$=\int\limits_{\dot E}G(r)\,d\bar{\bar{\mu}}(q)=G(r).$$

Hence and from (8) we obtain (2).

Remark. Let

$$C_n(r)\overset{\mathrm{df}}{=}\inf_{p^{(n)}\subset E}\Big\{\max_{(j)}\prod_{\substack{k=0\\k\neq j}}^{n}\frac{\omega(r,p_k)}{\omega(p_j,p_k)}\Big\}.$$

From (1) it follows that

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log C_n(r)\leqslant G(r). \tag{9}$$

Let $\bar{\bar{\bar{q}}}^{(n)}$, $\bar{\bar{\bar{\mu}}}_n$, n_k and $\bar{\bar{\bar{\mu}}}$ be defined analogically as in the proof of (2). Then we can prove similarly as before that for p belonging to the support of $\bar{\bar{\bar{\mu}}}$

$$\lim_{k\to\infty}\frac{1}{n_k}\log C_{n_k}(r)\geqslant\int\limits_E\log\frac{1}{\omega(p,q)}\,d\bar{\bar{\bar{\mu}}}(q)-\int\limits_E\log\frac{1}{\omega(q,r)}\,d\bar{\bar{\bar{\mu}}}(q).$$

Integrating now this inequality with respect to $\bar{\mu}$ we obtain by (4) and (5)

$$\lim_{k\to\infty}\frac{1}{n_k}\log C_{n_k}(r)\geqslant\int\limits_{\dot E}\Big[\int\limits_{\dot E}\log\frac{1}{\omega(p,q)}\,d\bar{\mu}(p)-\log\frac{1}{\omega(q,r)}\Big]\,d\bar{\mu}(q)$$

$$=\int\limits_E G(r)\,d\bar{\bar{\bar{\mu}}}(q)=G(r).$$

Hence and from (9) we get the equation

$$\lim_{n\to\infty}\frac{1}{n}\log C_n(r)=G(r).$$

This result was also obtained on another way by F. Leja (see [5] and also [4], p. 267) and by A. Szybiak [6].

REFERENCES

[1] W. Bach, *A simple proof of the uniqueness of the extremal measure*, Annales Polonici Mathematici 12 (1962), p. 207-208.

[2] — *A solution of the problem of four limits*, ibidem (to appear).

[3] O. Frostmann, *Potentiel d'équilibre et capacité des ensembles*, Thèse, Lund 1935.

[4] F. Leja, *Teoria funkcji analitycznych*, Warszawa 1957.

[5] — *Sur les suites de polynômes, les ensembles fermés et la fonction de Green*, Annales de la Société Polonaise de Mathématique 12 (1934), p. 57-71,

[6] A. Шибяк, *Об одной постановке главной задачи в теории обобщённых потенциалов*, Annales Polonici Mathematici 13 (1963), p. 57-66.

Reçu par la Rédaction le 18. 6. 1963