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A GENERALIZATION OF SCHWARZ'S LEMMA AND OF HADAMARD'S THREE CIRCLES THEOREM

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1. Introduction. In section 2 of this note a generalization of the classical lemma of Schwarz as well as its extension to the case of several complex variables are given. From these results the inequalities of Schwarz's lemma type given in [2] and [1] follow as particular cases. As an application of the extension a new proof of the Poincaré theorem on the impossibility of biholomorphic transformation of a polycylinder onto a ball in C^n $(n \geqslant 2)$ are given.

Section 3 is devoted to a generalization of Hadamard's three circles theorem. The three spheres theorem proved in [2], Chapter III, follows as a particular case from this generalization.

2. Generalization of the lemma of Schwarz. Let $S(z) = S(z_1, \ldots z_n)$ be a real uppersemicontinuous absolutely homogeneous function of degree 1, i. e.

$$(1) \qquad \limsup_{z \to z^0} S(z) \leqslant S(z^0) \quad \text{and} \quad S(\lambda z) = |\lambda| S(z),$$

where z, z^0 are arbitrary points of C^n and λ is any complex number. Let, moreover, $S(z) > 0$, if $z \neq 0 = (0, \ldots, 0)$. Under these assumptions

$$(2) \qquad D = \{z : S(z) < 1\}$$

is a domain such that if $z^0 \in D$, then $\{z : z_k = \lambda z_k^0, k = 1, \ldots, n, |\lambda| \leqslant 1\} \subset D$, i. e. D is a circular domain containing its origin 0. Therefore any function $f(z)$ holomorphic in D may be developped in a series of homogeneous polynomials

$$f(z) = \sum_{l=0}^{\infty} P_l(z), \qquad z \in D,$$

uniformly convergent on every compact subset of D (see [3]). We shall say that $f(z)$ has a *zero of order* ν *at* 0, if $P_0(z) \equiv \ldots \equiv P_{\nu-1}(z) \equiv 0$.

Let $T(w) = T(w_1, \ldots, w_m)$ be a non-negative plurisubharmonic function in C^m such that

$$(3)\qquad T\left(\frac{1}{\lambda}w\right) \geqslant \frac{1}{|\lambda|}T(w) \quad \text{for} \quad w \in C^m \text{ and } |\lambda| \leqslant 1,$$

$$(4)\qquad T(\lambda w) = T(w) \quad \text{for} \quad w \in C^m \text{ and } |\lambda| = 1.$$

LEMMA. *Suppose functions $w_k = f_k(z)$ $(k = 1, \ldots, m)$ are holomorphic in D and have zero of order ν at 0. Let*

$$(5)\qquad T(f_1(z), \ldots, f_m(z)) \leqslant M, \quad z \in D.$$

Then

$$(6)\qquad T(f_1(z), \ldots, f_m(z)) \leqslant M[S(z)]^\nu, \quad z \in D.$$

Proof. Observe that because of (3) we have $|\lambda| T(0) \geqslant T(0)$ for $|\lambda| \leqslant 1$, whence $T(0) = 0$. Thus (6) is true for $z = 0$. Suppose $z^* \in D$, $z^* \neq 0$. Then there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then $S(z^*) < 1-\varepsilon$. Put

$$z^0 = \frac{1-\varepsilon}{S(z^*)} z^*.$$

We have $z^0 \in D$, because $S(z^0) = 1-\varepsilon < 1$. The function $f_k(z)$ having zero of order ν at 0, the function

$$\frac{1}{\lambda^\nu} f_k(\lambda z^0)$$

is holomorphic for $|\lambda| \leqslant 1$. Thus the function

$$T\left(\frac{1}{\lambda^\nu} f_1(\lambda z^0), \ldots, \frac{1}{\lambda^\nu} f_m(\lambda z^0)\right)$$

is subharmonic for $|\lambda| \leqslant 1$. Because of (4) and (5) we have

$$T\left(\frac{1}{\lambda^\nu} f_1(\lambda z^0), \ldots, \frac{1}{\lambda^\nu} f_m(\lambda z^0)\right) = T(f_1(\lambda z^0), \ldots, f_m(\lambda z^0)) \leqslant M$$

for $|\lambda| = 1$. Owing to this and applying the maximum principle for subharmonic functions we get

$$T\left(\frac{1}{\lambda^\nu} f_1(\lambda z^0), \ldots, \frac{1}{\lambda^\nu} f_m(\lambda z^0)\right) \leqslant M, \quad |\lambda| \leqslant 1.$$

Hence, in view of (3),

$$T(f_1(\lambda z^0), \ldots, f_m(\lambda z^0)) \leqslant M|\lambda|^\nu \quad \text{for} \quad |\lambda| \leqslant 1,$$

whence by putting $\lambda = S(z^*)/(1-\varepsilon)$ we get in virtue of the equation

$$z^0 = \frac{1-\varepsilon}{S(z^*)} z^*$$

the inequality

$$T(f_1(z^*), \ldots, f_m(z^*)) \leqslant M\left[\frac{S(z^*)}{1-\varepsilon}\right]^\nu.$$

In view of the arbitrariness of $z^* \in D$ and $\varepsilon > 0$ the proof is completed.

COROLLARY 2.1. *If $m = n = 1$, $S(z) = |z|$, $T(w) = |w|$, then the Lemma reduces to the well known classical lemma of Schwarz for functions of one complex variable.*

COROLLARY 2.2. *One may easily check that the function*

$$T(w) = \left(\sum_1^m |w_k|^{p_k}\right)^{1/p},$$

where $p_k > 0$ and $p = \min p_k$ is plurisubharmonic in C^m and satisfies (3) *and* (4). *Thus by putting*

$$S(z) \equiv ||z|| = \left(\sum_{k=1}^n |z_k|^2\right)^{\frac{1}{2}},$$

(6) *yields a generalization of the inequalities proved in* [1] *and* [2].

COROLLARY 2.3. *Let D and G be bounded circular domains of holomorphy in C^n with their centers at 0. It is known* [4] *that there exist absolutely homogeneous functions $S(z)$ and $T(z)$ plurisubharmonic in C^n such that*

$$D = \{z\colon S(z) < 1\} \quad \textit{and} \quad G = \{z\colon T(z) < 1\}.$$

As an immediate consequence of the Lemma we get the following

THEOREM. *If*

$$(7)\qquad w_k = f_k(z), \quad k = 1, \ldots, n,$$

is a biholomorphic transformation of D onto G such that $f_k(0) = 0$ $(k = 1, \ldots, n)$, then

$$(8)\qquad T(f_1(z), \ldots, f_n(z)) = S(z), \quad z \in D.$$

The geometrical meaning of (8) is that the level hypersurface $S(z) = r$, where $0 < r < 1$, is mapped by transformation (7) onto the level hypersurface $T(z) = r$.

COROLLARY 2.4 (Theorem of Poincaré). *There is no biholomorphic transformation of a polycylinder*

$$P = \left\{z\colon \max\left(\frac{|z_1|}{\varrho_1}, \ldots, \frac{|z_n|}{\varrho_n}\right) < 1\right\}$$

onto a ball $B = \{z\colon ||z||/\varrho < 1\}$ in C^n, $n \geqslant 2$.

Proof. Suppose

(9) $$w_k = f_k(z), \quad k = 1, \ldots, n,$$

is a biholomorphic transformation of P onto B. Without loss of generality we may assume that $f_k(0) = 0$, $k = 1, \ldots, n$. Then a simple application of (8) gives

$$\max\left(\frac{|f_1(z)|}{\varrho_1}, \ldots, \frac{|f_n(z)|}{\varrho_n}\right) = \frac{||z||}{\varrho}, \quad \text{if} \quad \frac{||z||}{\varrho} < 1.$$

This means that the boundary of the ball $B_r = \{z: ||z||/\varrho < r\}$, $0 < r < \varrho$, is mapped onto the boundary of the polycylinder $P_r = \{z: \max(|z_1|/\varrho_1, \ldots, |z_n|/\varrho_n) < r\}$. Let $z^0 = (z_1^0, \ldots, z_n^0)$ be an arbitrary fixed point of B such that $\|z^0\|/\varrho = r$ and $|f_1(z^0)|/\varrho_1 < r$ (but $\max(|f_2(z^0)|/\varrho_2, \ldots, |f_n(z^0)|/\varrho_n) = r$.) The function $F(z) = 1/[2-(z_1\bar{z}_1^0 + \ldots + z_n\bar{z}_n^0)(r\varrho)^{-2}]$ is holomorphic in the closure $\bar{B}_r$ of B_r and

$$|F(z)| < 1 = F(z^0) \quad \text{for} \quad z \in \bar{B}_r, \quad z \neq z^0.$$

Thus the function $F^*(W) = F(g_1(w), \ldots, g_n(w))$, where $z_k = g_k(w)$ $(k = 1, \ldots, n)$ is the transformation converse to (9), is holomorphic in $\bar{P}_r$ and

$$|F^*(w)| < 1 = F(w^0), \quad w \in \bar{P}_r, \quad w \neq w^0,$$

where $w^0 = (f_1(z^0), \ldots, f_n(z^0))$. This is impossible, since $\max(|F^*(w)|, w \in \bar{P}_r) = \max(|F^*(w)|, w \in \{w: |w_k|/\varrho_k = r, k = 1, \ldots, n\})$. The proof is thus completed.

Observe that we have reduced the proof of Poincaré's theorem to the statement which says that a necessary condition that domains D and G could be biholomorphically mapped one onto each other by a transformation topological in the closures of the domains is that the Bergman-Šilov boundary of D is mapped onto the Bergman-Šilov boundary of G under this transformation.

This statement along with equation (8) enables us to show in many other cases that two given Reinhardt's circular domains are not biholomorphically transformable one onto each other.

So it seems worthwhile to investigate the following problem. Let D and G be bounded domains in C^n, $n \geqslant 2$. Suppose the closures $\bar{D}$ and $\bar{G}$ are homeomorphic but the Bergman-Šilov boundaries $S(D)$ and $S(G)$ of D and G, respectively, are not. Is it then true that there is no biholomorphic transformation of D onto G? (**P 449**)

3. Generalization of Hadamard's three circles theorem. Let $S(z) = S(z_1, \ldots, z_n)$ be a continuous real function defined in C^n $(n \geqslant 1)$ such that $S(z) > 0$ for $z \neq 0$ and

$$S(\lambda z) = |\lambda| S(z) \quad \text{for} \quad z \in C^n, \ |\lambda| < \infty.$$

Let $H(w) = H(w_1, \ldots, w_m)$ be a non-negative real function defined in a domain $D \subset C^m$ such that $\log H(w)$ is plurisubharmonic in D.

THEOREM. *Suppose functions $f_1(z), \ldots, f_m(z)$ are holomorphic in $G = \{z: r_1 \leqslant S(z) \leqslant r_3\}$ and, moreover, $f(z) = (f_1(z), \ldots, f_m(z)) \in D$, if $z \in G$. Let $0 < r_1 < r_2 < r_3$ and put*

$$M_k = \max_{S(z)=r_k} H(f(z)), \quad k = 1, 2, 3.$$

Then

(10) $$M_2^{\alpha_2} \leqslant M_1^{\alpha_1} M_3^{\alpha_3},$$

where $\alpha_1 = \log(r_3/r_2)$, $\alpha_2 = \log(r_3/r_1)$, $\alpha_3 = \log(r_2/r_1)$, i. e. the function $\log M(\varrho)$, where $M(\varrho) = \max_{S(z)=\varrho} H(f(z))$, is convex with respect to $\log\varrho$ for $r_1 \leqslant \varrho \leqslant r_3$.

Proof. We shall reduce the proof of (10) to the proof of classical Hadamard's three circles theorem. Namely let us observe that

$$\{z: S(z) = \varrho\} = \{z: z = \lambda a, \ S(a) = 1, \ |\lambda| = \varrho\}$$

and that the function $g(\lambda, a) = |\lambda|^{\alpha} H(f(\lambda a))$, a being arbitrary real number, is subharmonic in the annulus $r_1 \leqslant \lambda \leqslant r_3$ for each fixed a such that $S(a) = 1$. Hence, by the maximum principle for subharmonic functions, we have

$$g(\lambda, a) \leqslant \max(r_1^{\alpha} M_1, r_3^{\alpha} M_3)$$

for $r_1 \leqslant |\lambda| \leqslant r_3$ and $S(a) = 1$. Thus $r_2^{\alpha} M_2 \leqslant \max(r_1^{\alpha} M_1, r_3^{\alpha} M_3)$, whence $M_2 \leqslant \max(r_1^{\alpha} r_2^{-\alpha} M_1, r_3^{\alpha} r_2^{-\alpha} M_3)$. For $\alpha = -(\log M_3/M_1)/(\log r_3/r_1)$ this implies (10).

COROLLARY 3.1. *If $m = n = 1$, $H(w) = |w|$, $D = C^1$, and $S(z) = |z|$, then* (10) *provides the classical Hadamard's three circles theorem.*

COROLLARY 3.2. *One may check that if $H(w) = \left(\sum_{k=1}^{m} |w_k|^{p_k}\right)^{1/p}$, where $p_k \geqslant 0$, $p > 0$, then the function $\log H(w)$ is plurisubharmonic in C^m. Thus by putting $S(z) = ||z||$ we see that* (10) *yields as a particular case the "three spheres theorem" proved in* [2].

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